Lattice QCD constraints on the QCD critical point

Alexei Bazavov
Michigan State University
June 2, 2018
Introduction

QCD phase diagram
Lattice gauge theory
Challenges

Results at $\mu_B = 0$
Chiral symmetry restoration

Results at $\mu_B > 0$
Curvature of the crossover line
The equation of state at $O(\mu_B^6)$
Freeze-out parameters
Constraints on the critical point

Conclusion
I Study response of the system to change of external parameters, i.e. temperature and baryon density, asymptotic freedom suggests a weakly interacting phase.

I Experimental program: RHIC, LHC, FAIR, NICA

I RHIC BES: search for the critical point

I First-principle calculations are possible at $\mu_B/T = 0$, expansions/extrapolations at small $\mu_B/T$.

---

1Collins, Perry (1975), Cabbibo, Parisi (1975)
Study response of the system to change of external parameters, i.e. temperature and baryon density, asymptotic freedom suggests a weakly interacting phase\(^1\)

\(^1\)Collins, Perry (1975), Cabbibo, Parisi (1975)
Study response of the system to change of external parameters, i.e. temperature and baryon density, asymptotic freedom suggests a weakly interacting phase \(^1\)

Experimental program: RHIC, LHC, FAIR, NICA

\(^1\) Collins, Perry (1975), Cabbibo, Parisi (1975)
Study response of the system to change of external parameters, i.e. temperature and baryon density, asymptotic freedom suggests a weakly interacting phase\(^1\)

Experimental program: RHIC, LHC, FAIR, NICA

RHIC BES: search for the critical point

\(^1\)Collins, Perry (1975), Cabbibo, Parisi (1975)
Study response of the system to change of external parameters, i.e. temperature and baryon density, asymptotic freedom suggests a weakly interacting phase\(^1\)

Experimental program: RHIC, LHC, FAIR, NICA

RHIC BES: search for the critical point

First-principle calculations are possible at $\mu_B/T = 0$, expansions/extrapolations at small $\mu_B/T$

\(^1\)Collins, Perry (1975), Cabbibo, Parisi (1975)
Lattice gauge theory

- Start with the path integral quantization, Euclidean signature:

\[
\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}[\psi] \mathcal{D}[\psi] \mathcal{D}[A] \, \mathcal{O} \exp(-S_E(T, V, \bar{\mu})),
\]

\[
\mathcal{Z}(T, V, \bar{\mu}) = \int \mathcal{D}[\psi] \mathcal{D}[\psi] \mathcal{D}[A] \exp(-S_E(T, V, \bar{\mu})),
\]
Lattice gauge theory

- Start with the path integral quantization, Euclidean signature:

\[
\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}[\psi] \mathcal{D}[\bar{\psi}] \mathcal{D}[A] \mathcal{O} \exp(-S_E(T, V, \bar{\mu})),
\]

\[
\mathcal{Z}(T, V, \bar{\mu}) = \int \mathcal{D}[\psi] \mathcal{D}[\bar{\psi}] \mathcal{D}[A] \exp(-S_E(T, V, \bar{\mu})),
\]

\[
S_E(T, V, \bar{\mu}) = -\int_0^{1/T} d\chi_0 \int_\mathcal{V} d^3x \mathcal{L}^E(\bar{\mu}),
\]
Lattice gauge theory

- Start with the path integral quantization, Euclidean signature:

\[
\langle \mathcal{O} \rangle = \frac{1}{Z} \int D[\psi] D[\bar{\psi}] D[A] \mathcal{O} \exp(-S_E(T, V, \bar{\mu})),
\]

\[
Z(T, V, \bar{\mu}) = \int D[\psi] D[\bar{\psi}] D[A] \exp(-S_E(T, V, \bar{\mu})),
\]

\[
S_E(T, V, \bar{\mu}) = -\int_0^{1/T} d\chi_0 \int d^3x \mathcal{L}_E^{\mathcal{E}}(\bar{\mu}),
\]

\[
\mathcal{L}_E^{\mathcal{E}}(\bar{\mu}) = \mathcal{L}_{QCD}^{\mathcal{E}} + \sum_{f=u,d,s} \mu_f \bar{\psi}_f \gamma_0 \psi_f
\]
Lattice gauge theory

- Start with the path integral quantization, Euclidean signature:

\[
\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}[\psi] \mathcal{D}[\bar{\psi}] \mathcal{D}[A] \mathcal{O} \exp(-S_E(T, V, \bar{\mu})),
\]

\[
Z(T, V, \bar{\mu}) = \int \mathcal{D}[\psi] \mathcal{D}[\bar{\psi}] \mathcal{D}[A] \exp(-S_E(T, V, \bar{\mu})),
\]

\[
S_E(T, V, \bar{\mu}) = -\int_0^{1/T} d\chi_0 \int_0^V d^3x \mathcal{L}^E(\bar{\mu}),
\]

\[
\mathcal{L}^E(\bar{\mu}) = \mathcal{L}_{QCD}^E + \sum_{f=u,d,s} \mu_f \bar{\psi}_f \gamma_0 \psi_f
\]

- Introduce a (non-perturbative!) regulator – minimum space-time “resolution” scale \( a \), i.e. lattice, Wilson (1974)
Lattice gauge theory

- Start with the path integral quantization, Euclidean signature:

\[
\langle O \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}[\psi] \mathcal{D}[\bar{\psi}] \mathcal{D}[A] \, O \exp(-S_E(T, V, \bar{\mu})),
\]

\[
\mathcal{Z}(T, V, \bar{\mu}) = \int \mathcal{D}[\psi] \mathcal{D}[\bar{\psi}] \mathcal{D}[A] \exp(-S_E(T, V, \bar{\mu})),
\]

\[
S_E(T, V, \bar{\mu}) = -\int_0^{1/T} d\chi_0 \int_V d^3x \mathcal{L}^E(\bar{\mu}),
\]

\[
\mathcal{L}^E(\bar{\mu}) = \mathcal{L}^E_{QCD} + \sum_{f=u,d,s} \mu_f \bar{\psi}_f \gamma_0 \psi_f
\]

- Introduce a (non-perturbative!) regulator – minimum space-time “resolution” scale \( a \), i.e. lattice, Wilson (1974)

- The lattice spacing \( a \) acts as a UV cutoff, \( p_{max} \sim \pi/a \)
Lattice gauge theory

- Start with the path integral quantization, Euclidean signature:

\[
\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}[\psi] \mathcal{D}[\bar{\psi}] \mathcal{D}[\mathcal{A}] \mathcal{O} \exp(-S_E(T, V, \bar{\mu})),
\]

\[
\mathcal{Z}(T, V, \bar{\mu}) = \int \mathcal{D}[\psi] \mathcal{D}[\bar{\psi}] \mathcal{D}[\mathcal{A}] \exp(-S_E(T, V, \bar{\mu})),
\]

\[
S_E(T, V, \bar{\mu}) = -\frac{1}{T} \int_0^{1/T} dx_0 \int_V d^3x \mathcal{L}^E(\bar{\mu}),
\]

\[
\mathcal{L}^E(\bar{\mu}) = \mathcal{L}_{QCD}^E + \sum_{f=u,d,s} \mu_f \bar{\psi}_f \gamma_0 \psi_f
\]

- Introduce a (non-perturbative!) regulator – minimum space-time “resolution” scale \( a \), i.e. lattice, Wilson (1974)

- The lattice spacing \( a \) acts as a UV cutoff, \( p_{\text{max}} \sim \pi/a \)

- The integrals can be evaluated with importance sampling methods
Challenges

- Broken symmetries – e.g., Lorentz, chiral
Challenges

- Broken symmetries – e.g., Lorentz, chiral
- Fermion doubling

\[
\begin{align*}
Z &= Z_D[U]\ D[S\ G[U]_S]\ [U]\ \\
&= Z_D[U]\ e^{S_G[U]}\ \det|M[U]| \equiv \mathcal{Z}_{2}\ .
\end{align*}
\]

The effective action is highly non-local, Monte Carlo sampling is costly.
The computational cost is determined by the condition number of the fermion matrix, which scales with the inverse lightest quark mass.

Sign problem at $\mu_B > 0$.
Real-time properties are hard to access.
Challenges

- Broken symmetries – e.g., Lorentz, chiral
- Fermion doubling
- Grassmann fields (fermions) cannot be sampled, integrate them out:

\[
\mathcal{Z} = \int \mathcal{D}[U] \mathcal{D}[\psi] \mathcal{D}[\bar{\psi}] e^{-S_G[U] - S_F[\bar{\psi}, \psi, U]}
= \int \mathcal{D}[U] e^{-S_G[U]} \det |M[U]| \]
Challenges

- Broken symmetries – e.g., Lorentz, chiral
- Fermion doubling
- Grassmann fields (fermions) cannot be sampled, integrate them out:

\[
\mathcal{Z} = \int \mathcal{D}[U] \mathcal{D}[\psi] \mathcal{D}[\bar{\psi}] e^{-S_G[U] - S_F[\bar{\psi}, \psi, U]}
\]

\[
= \int \mathcal{D}[U] e^{-S_G[U]} \det |M[U]| 
\]

- The effective action is highly non-local, Monte Carlo sampling is costly
- The computational cost is determined by the condition number of the fermion matrix, which scales with the inverse lightest quark mass
Challenges

- Broken symmetries – e.g., Lorentz, chiral
- Fermion doubling
- Grassmann fields (fermions) cannot be sampled, integrate them out:

\[
Z = \int \mathcal{D}[U]\mathcal{D}[\psi]\mathcal{D}[\bar{\psi}] e^{-S_G[U]-S_F[\bar{\psi},\psi,U]} \\
= \int \mathcal{D}[U] e^{-S_G[U]} \det |M[U]| 
\]

- The effective action is highly non-local, Monte Carlo sampling is costly
- The computational cost is determined by the condition number of the fermion matrix, which scales with the inverse lightest quark mass
- Sign problem at $\mu_B > 0$
Challenges

- Broken symmetries – e.g., Lorentz, chiral
- Fermion doubling
- Grassmann fields (fermions) cannot be sampled, integrate them out:

\[
Z = \int \mathcal{D}[U] \mathcal{D}[\psi] \mathcal{D}[\bar{\psi}] e^{-S_G[U]} - S_F[\bar{\psi}, \psi, U]
\]

\[
= \int \mathcal{D}[U] e^{-S_G[U]} \det |M[U]|
\]

- The effective action is highly non-local, Monte Carlo sampling is costly
- The computational cost is determined by the condition number of the fermion matrix, which scales with the inverse lightest quark mass
- Sign problem at \( \mu_B > 0 \)
- Real-time properties are hard to access
How to access $\mu_B > 0$?

Method 1: Taylor expansion (Allton et al. (2002)), evaluate various derivatives at $\mu = 0$, e.g.

$$u_2 = \text{Tr} D_{\mu} M_1 u_{M_0} u (M_1 u_M) + \text{Tr} (M_1 u_M)$$

Method 2: Perform simulations at imaginary chemical potential, then evaluate the derivatives of $P(i\mu)$ (Lombardo (1999), de Forcrand, Philipsen (2002))

Methods 3, 4, ...: Complex Langevin dynamics, contour deformation, reweighting/density of states, ...
How to access $\mu_B > 0$?

- **Method 1**: Taylor expansion (Allton et al. (2002)), evaluate various derivatives at $\mu = 0$, e.g.

\[
\chi^u_2 = \frac{T}{V} \left\langle \text{Tr} \left( M_u^{-1} M''_u - (M_u^{-1} M'_u)^2 \right) + \left( \text{Tr}(M_u^{-1} M'_u) \right)^2 \right\rangle
\]
How to access $\mu_B > 0$?

- **Method 1**: Taylor expansion (Allton et al. (2002)), evaluate various derivatives at $\mu = 0$, e.g.

  $$
  \chi_2^u = \frac{T}{V} \left< \text{Tr} \left( M_u^{-1} M''_u - (M_u^{-1} M'_u)^2 \right) + \left( \text{Tr}(M_u^{-1} M'_u) \right)^2 \right>
  $$

- **Method 2**: Perform simulations at imaginary chemical potential, then evaluate the derivatives of $P(i\mu)$ (Lombardo (1999), de Forcrand, Philipsen (2002))
How to access $\mu_B > 0$?

- **Method 1**: Taylor expansion (Allton et al. (2002)), evaluate various derivatives at $\mu = 0$, e.g.
  \[
  \chi_2^u = \frac{T}{V} \left\langle \text{Tr} \left( M_u^{-1} M''_u - (M_u^{-1} M'_u)^2 \right) + \left( \text{Tr}(M_u^{-1} M'_u) \right)^2 \right\rangle
  \]

- **Method 2**: Perform simulations at imaginary chemical potential, then evaluate the derivatives of $P(i\mu)$ (Lombardo (1999), de Forcrand, Philipsen (2002))

- **Methods 3, 4, ...**: Complex Langevin dynamics, contour deformation, reweighting/density of states, ...
Method 1: Taylor expansion

- The chemical potentials for conserved charges $B$, $Q$, $S$:

\[
\begin{align*}
\mu_u &= \frac{1}{3} \mu_B + \frac{2}{3} \mu_Q, \\
\mu_d &= \frac{1}{3} \mu_B - \frac{1}{3} \mu_Q, \\
\mu_s &= \frac{1}{3} \mu_B - \frac{1}{3} \mu_Q - \mu_s
\end{align*}
\]
Method 1: Taylor expansion

The chemical potentials for conserved charges $B$, $Q$, $S$:

\[
\begin{align*}
\mu_u &= \frac{1}{3}\mu_B + \frac{2}{3}\mu_Q, \\
\mu_d &= \frac{1}{3}\mu_B - \frac{1}{3}\mu_Q, \\
\mu_s &= \frac{1}{3}\mu_B - \frac{1}{3}\mu_Q - \mu_S.
\end{align*}
\]

The pressure can be expanded in Taylor series

\[
\frac{P}{T^4} = \frac{1}{VT^3} \ln \mathcal{Z}(T, V, \hat{\mu}_u, \hat{\mu}_d, \hat{\mu}_s) = \sum_{i,j,k=0}^{\infty} \frac{\chi_{ijk}^{BQS}}{i!j!k!} \hat{\mu}_B^i \hat{\mu}_Q^j \hat{\mu}_S^k
\]
Method 1: Taylor expansion

- The chemical potentials for conserved charges $B$, $Q$, $S$:

\[
\mu_u = \frac{1}{3} \mu_B + \frac{2}{3} \mu_Q,
\]
\[
\mu_d = \frac{1}{3} \mu_B - \frac{1}{3} \mu_Q,
\]
\[
\mu_s = \frac{1}{3} \mu_B - \frac{1}{3} \mu_Q - \mu_S
\]

- The pressure can be expanded in Taylor series

\[
\frac{P}{T^4} = \frac{1}{VT^3} \ln \mathcal{Z}(T, V, \hat{\mu}_u, \hat{\mu}_d, \hat{\mu}_s) = \sum_{i,j,k=0}^{\infty} \frac{\chi_{ijk}^{BQS}}{i! j! k!} \hat{\mu}_B^i \hat{\mu}_Q^j \hat{\mu}_S^k
\]

- The generalized susceptibilities are evaluated at vanishing chemical potential

\[
\chi_{ijk}^{BQS} \equiv \chi_{ijk}^{BQS}(T) = \frac{\partial P(T, \hat{\mu})/T^4}{\partial \hat{\mu}_B^i \partial \hat{\mu}_Q^j \partial \hat{\mu}_S^k} \bigg|_{\hat{\mu}=0}, \quad \hat{\mu} \equiv \frac{\mu}{T}
\]
Fluctuations of conserved charges

Strangeness (left) and baryon number (right) fluctuations
The number densities can also be represented with Taylor expansions:

\[
\frac{n_X}{T^3} = \frac{\partial P}{\partial \mu_X}, \quad X = B, Q, S
\]
Constrained series expansions

- The number densities can also be represented with Taylor expansions:

\[
\frac{n_X}{T^3} = \frac{\partial P}{\partial \hat{\mu}_X} T^4, \quad X = B, Q, S
\]

- In heavy-ion collisions there are additional constraints:

\[
n_S = 0, \quad \frac{n_Q}{n_B} = 0.4
\]
Constrained series expansions

- The number densities can also be represented with Taylor expansions:

\[
\frac{n_X}{T^3} = \frac{\partial P}{T^4 \partial \hat{\mu}_X}, \quad X = B, Q, S
\]

- In heavy-ion collisions there are additional constraints:

\[
n_S = 0, \quad \frac{n_Q}{n_B} = 0.4
\]

- These constraints can be fulfilled by

\[
\begin{align*}
\hat{\mu}_Q(T, \mu_B) &= q_1(T)\hat{\mu}_B + q_3(T)\hat{\mu}_B^3 + q_5(T)\hat{\mu}_B^5 + \ldots , \\
\hat{\mu}_S(T, \mu_B) &= s_1(T)\hat{\mu}_B + s_3(T)\hat{\mu}_B^3 + s_5(T)\hat{\mu}_B^5 + \ldots
\end{align*}
\]
Method 2: Imaginary chemical potential\textsuperscript{2}

\[ \frac{d(p/T^4)}{d\mu} \]

\( T_c(\mu) \)

\text{continuation}

\( \mu^2/T^2 \)

\( \kappa \)

\text{lattice simulations} \hspace{1cm} \text{real chemical potentials}

\text{Roberge-Weiss}

\text{many exploratory studies:}

\[ \text{[de Forcrand & Philipsen hep-lat/0205016]} \]
\[ \text{[Philipsen 0708.1293]} \]
\[ \text{[Philipsen 1402.0838]} \]
\[ \text{[Cea et al hep-lat/0612018,0905.1292,1202.5700]} \]

\text{Figure from the talk at Quark Matter 2018 by S. Borsanyi}

\textsuperscript{2}Figure from the talk at Quark Matter 2018 by S. Borsanyi
Baryon number susceptibilities

$\chi_2^B$, $\chi_4^B$, $\chi_6^B$, $\chi_8^B$ vs $T/\text{MeV}$. 

$^{3}$Borsanyi et al. [WB], 1805.04445
Results at $\mu_B = 0$
Chiral symmetry restoration

- Chiral condensate and susceptibility

\[
\langle \bar{\psi}\psi \rangle_f = \frac{T}{V} \frac{\partial \ln Z}{\partial m_f}, \quad \chi(T) = \frac{\partial \langle \bar{\psi}\psi \rangle_f}{\partial m_f}
\]
Chiral symmetry restoration

- Chiral condensate and susceptibility

\[ \langle \bar{\psi} \psi \rangle_f = \frac{T}{V} \frac{\partial \ln Z}{\partial m_f}, \quad \chi(T) = \frac{\partial \langle \bar{\psi} \psi \rangle_f}{\partial m_f} \]
Chiral symmetry restoration

- Chiral condensate and susceptibility

\[ \langle \bar{\psi} \psi \rangle_f = \frac{T}{V} \frac{\partial \ln Z}{\partial m_f}, \quad \chi(T) = \frac{\partial \langle \bar{\psi} \psi \rangle_f}{\partial m_f} \]

The chiral crossover temperature at \( \mu_B = 0 \) (Borsanyi et al. [BW] (2010), Bazavov et al. [HotQCD] (2012))

\[ T_c = 154 \pm 9 \text{ MeV} \]
Chiral symmetry restoration (update)\textsuperscript{4}

The chiral crossover temperature at $\mu_B = 0$ (HotQCD, preliminary)

$$T_c = 156.5 \pm 1.5 \text{ MeV}$$

\textsuperscript{4}Figure from the talk at Quark Matter 2018 by P. Steinbrecher
Chiral symmetry restoration (update)\textsuperscript{5}

- Comparison with earlier results

![Graph showing comparison with earlier results](image)

\textsuperscript{5}Figure from talk at Quark Matter 2018 by P. Steinbrecher
Results at $\mu_B > 0$
Curvature of the chiral crossover line

- Change in the chiral crossover temperature with $\mu_B$

\[
\frac{T_c(\mu_B)}{T_c(0)} = 1 - \kappa_2 \left( \frac{\mu_B}{T_c(0)} \right)^2 - \kappa_4 \left( \frac{\mu_B}{T_c(0)} \right)^4 + O(\mu_B^6)
\]

\[\text{Figure from the talk at Quark Matter 2018 by M. D’Elia}\]
Chiral crossover at $\mu_B > 0$

The magnitude of the chiral susceptibility shows almost no change with increasing $\mu_B > 0$.

No indication that the crossover is getting stronger.

Similar conclusion from the baryon number fluctuations along the crossover line.

---

Figure from the talk at Quark Matter 2018 by P. Steinbrecher
Chiral crossover at $\mu_B > 0$

The magnitude of the chiral susceptibility shows almost no change with increasing $\mu_B > 0$.
Chiral crossover at $\mu_B > 0$

The magnitude of the chiral susceptibility shows almost no change with increasing $\mu_B > 0$

No indication that the crossover is getting stronger

---

Figure from the talk at Quark Matter 2018 by P. Steinbrecher
The magnitude of the chiral susceptibility shows almost no change with increasing $\mu_B > 0$.

No indication that the crossover is getting stronger.

Similar conclusion from the baryon number fluctuations along the crossover line.

---

Figure from the talk at Quark Matter 2018 by P. Steinbrecher

---

May 16, 2018

Patrick Steinbrecher

Slide 20
The equation of state at $O(\mu_B^6)$

- The equation of state at $\mu_B = 0$

---

8 Borsanyi et al. [WB] (2014), Bazavov et al. [HotQCD] (2014)
The equation of state at $O(\mu_B^6)$

- The equation of state at $\mu_B = 0^8$

- Additional contribution at $\mu_B > 0$, $\mu_Q = \mu_S = 0$:

\[
\frac{\Delta P}{T^4} = \frac{1}{2} \frac{\chi_2^B(T)\hat{\mu}_B^2}{\chi_2^B(T)} \left( 1 + \frac{1}{12} \frac{\chi_4^B(T)\hat{\mu}_B^2}{\chi_2^B(T)} + \frac{1}{360} \frac{\chi_6^B(T)\hat{\mu}_B^4}{\chi_2^B(T)} + \ldots \right)
\]

\(^8\)Borsanyi et al. [WB] (2014), Bazavov et al. [HotQCD] (2014)
Bazavov et al. [HotQCD] (2017)
The equation of state at $O(\mu_B^6)$

- The contribution to the pressure due to finite chemical potential (left) and the baryon number density (right) for strangeness neutral systems:

$$n_S = 0, \quad \frac{n_Q}{n_B} = 0.4$$
Relativistic heavy-ion collisions

- Cumulants of the event-by-event multiplicity distributions:

\[
C_1 = \langle N \rangle, \quad C_2 = \langle (\delta N)^2 \rangle, \quad C_3 = \langle (\delta N)^3 \rangle, \quad C_4 = \langle (\delta N)^4 \rangle - 3 \langle (\delta N)^2 \rangle^2
\]
Relativistic heavy-ion collisions

- Cumulants of the event-by-event multiplicity distributions:
  
  \[ C_1 = \langle N \rangle, \quad C_2 = \langle (\delta N)^2 \rangle, \quad C_3 = \langle (\delta N)^3 \rangle, \quad C_4 = \langle (\delta N)^4 \rangle - 3 \langle (\delta N)^2 \rangle^2 \]

- Mean, variance, skewness and kurtosis:
  
  \[ M = C_1, \quad \sigma^2 = C_2, \quad S = \frac{C_3}{(C_2)^{3/2}}, \quad \kappa = \frac{C_4}{(C_2)^2} \]
Relativistic heavy-ion collisions

- Cumulants of the event-by-event multiplicity distributions:
  \[ C_1 = \langle N \rangle, \quad C_2 = \langle (\delta N)^2 \rangle, \quad C_3 = \langle (\delta N)^3 \rangle, \quad C_4 = \langle (\delta N)^4 \rangle - 3 \langle (\delta N)^2 \rangle^2 \]

- Mean, variance, skewness and kurtosis:
  \[ M = C_1, \quad \sigma^2 = C_2, \quad S = \frac{C_3}{(C_2)^{3/2}}, \quad \kappa = \frac{C_4}{(C_2)^2} \]
Freeze-out parameters

- Consider the ratios of cumulants:

\[
R_{31}^Q = \frac{S_Q \sigma_Q^3}{M_Q} = \frac{\chi_3^Q}{\chi_1^Q}, \quad R_{12}^Q = \frac{M_Q}{\sigma_Q^2} = \frac{\chi_1^Q}{\chi_2^Q}
\]

\footnote{Bazavov et al. [BNL-Bielefeld] (2012)}
Freeze-out parameters

- Consider the ratios of cumulants:
  \[ R_{31}^Q = \frac{S_Q\sigma_Q^3}{M_Q} = \frac{\chi_3^Q}{\chi_1^Q}, \quad R_{12}^Q = \frac{M_Q}{\sigma_Q^2} = \frac{\chi_1^Q}{\chi_2^Q} \]

- These ratios can be evaluated on the lattice for constrained system and serve as thermometer (left) and baryometer (right)\(^\text{10}\)

---

\(^{10}\)Bazavov et al. [BNL-Bielefeld] (2012)
Skewness and kurtosis

STAR: 0.4 GeV < p_T < 0.8 GeV
PRL 112 (2014) 032302

HRG

S_p \sigma_p^3 / M_p

M_p / \sigma_p^2

S_p \sigma_p^2 / M_p

K_p \sigma_p^2

S_p \sigma_p^3 / M_p

STAR: 0.4 GeV < p_T < 2.0 GeV
preliminary

M_p / \sigma_p^2

Freeze-out temperatures
[BNL-Bielefeld] (2017):
T_0 \approx 149 MeV for p_cut T = 0.8 GeV
T_0 = (153 \pm 5) MeV for p_cut T = 2.0 GeV
Skewness and kurtosis

Freeze-out temperatures
[BNL-Bielefeld] (2017):

\[ T_0 \leq 149 \text{ MeV} \]
for \( p_t^{\text{cut}} = 0.8 \text{ GeV} \)

\[ T_0 = (153 \pm 5) \text{ MeV} \]
for \( p_t^{\text{cut}} = 2.0 \text{ GeV} \)
Recent result by Borsanyi et al. [WB] 1805.04445
Constraints on the critical point

- For $\mu_Q = \mu_S = 0$ the net baryon-number susceptibility is

$$\chi_B^2(T, \mu_B) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \chi_{2n+2}^B \hat{\mu}_B^{2n}$$
Constraints on the critical point

- For $\mu_Q = \mu_S = 0$ the net baryon-number susceptibility is

$$\chi_2^B(T, \mu_B) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \chi_{2n+2}^B \hat{\mu}_B^{2n}$$

- The radius of convergence

$$r_{2n}^\chi = \sqrt{\frac{2n(2n-1)}{\chi_{2n+2}^B}}$$
**Constraints on the critical point**

- For $\mu_Q = \mu_S = 0$ the net baryon-number susceptibility is

  \[ \chi_2^B(T, \mu_B) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \chi_{2n+2}^B \hat{\mu}_B^{2n} \]

- The radius of convergence

  \[ r_{2n}^\chi \equiv \sqrt{\frac{2n(2n-1)\chi_{2n}^B}{\chi_{2n+2}^B}} \]

- We observe $\chi_6^B/\chi_4^B < 3$ for $135 < T < 155$ MeV $\Rightarrow r_4^\chi \geq 2$
Conclusion

- Lattice QCD calculations are now in the regime of the physical light quark masses and continuum limit is possible for many observables
- The most studied region of the QCD phase diagram is at $\mu_B = 0$
- At non-zero baryon chemical potential direct Monte Carlo simulations are not (yet) possible due to the sign problem
- The region of small $\mu/T$ can be explored with expansions in $\mu/T$ or by analytic continuation from imaginary $\mu$
- Generalized susceptibilities are now calculated up to 8th order in $\mu_B$
- The equation of state is now known up to the 6th order in $\mu_B$
- Ratios of the generalized susceptibilities can be related to experimentally measured cumulants of event-by-event multiplicity distributions
- Recent lattice calculations strongly disfavor QCD critical point in the region of $\mu_B < 2T$ in the temperature range $135 < T < 155$ MeV