Lattice QCD and the Proton Radius

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[1711.11385; Phys.Rev.D97.034504]

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Outline

- Nucleon form factors and radii on a lattice
- Results from the physical point
- Form factors at zero momentum
Basics of Hadron Structure in Lattice QCD

Lattice Field Theory $\leftrightarrow$ Numerical evaluation of the Path Integral

$$\langle q_x \bar{q}_y \ldots \rangle = \int \mathcal{D}(Glue) \int \mathcal{D}(Quarks) e^{-S_{\text{Glue}} - q(\not{D} + m)q} \left[ q_x \bar{q}_y \ldots \right]$$

Grassmann integration

$$= \int \mathcal{D}(Glue) e^{-S_{\text{Glue}}} \det(\not{D} + m) \left[ (\not{D} + m)^{-1} \right]_{x,y} \ldots$$

Hybrid Monte Carlo

Hadron Matrix Elements:

$$C^{\mathcal{O}}_{3\text{pt}}(T) = \langle N(T)\mathcal{O}(\tau)\bar{N}(0) \rangle = \sum_{n,m} Z_m e^{-E_n(T-\tau)} \langle n|\mathcal{O}|m \rangle e^{-E_m\tau} Z_n^*$$

"connected"

"disconnected"

$$\rightarrow Z_{00} e^{-M_{NT}} \left[ \langle P'|\mathcal{O}|P \rangle + \mathcal{O}\left(e^{-\Delta E_{10}T}, e^{-\Delta E_{10}\tau}, e^{-\Delta E_{10}(T-\tau)}\right) \right]$$

Ground state form factors

Systematic effects:
- excited states
- discretization errors
- finite volume
- unphysical (heavy) pion mass
Electric Form Factor

\[ \langle P + q | \bar{q} \gamma^\mu q | P \rangle = \tilde{U}_{P+q} \left[ F_1(Q^2) \gamma^\mu + F_2(Q^2) \frac{i\sigma^{\mu\nu} q_\nu}{2M_N} \right] U_P \]

\[ G_E(Q^2) = F_1(Q^2) - \frac{Q^2}{4M_N^2} F_2(Q^2) \]

Alberico et al parametrization
lattice data

\[ G_{Ep}(Q^2) \approx 1 - \frac{1}{6} Q^2 \langle r_E^2 \rangle^p + O(Q^4) \]

\[ \langle r^2 \rangle = -6 \frac{dG(Q^2)}{dQ^2} \bigg|_{Q^2=0} \]

Excited states are worse for \( m_\pi < 200 \text{ MeV} \)
Electric Form Factor at the Physical Point

\[ G_E(Q^2) = F_1(Q^2) - \frac{Q^2}{4M_N^2} F_2(Q^2) \]

\[ G_{Ev} = G_{Ep} - G_{En} \]

\[ m_\pi = 139 \text{ MeV} \quad a = 0.114 \text{ fm} \]

[RBC/LHP, in prep.]
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\[ m_\pi = 130 \text{ MeV} \quad a = 0.094 \text{ fm} \]

[1706.00469(ETMC)]

\[ t_s = 0.9 \text{ fm} \]

\[ t_s = 1.1 \text{ fm} \]

\[ t_s = 1.3 \text{ fm} \]

\[ t_s = 1.5 \text{ fm} \]

\[ t_s = 1.7 \text{ fm} \]

Summation, \( t_s \in [0.9, 1.7] \text{ fm} \)

Two-state, \( t_s \in [1.1, 1.7] \text{ fm} \)

Kelly parameterization

Sergey N. Syritsyn, for LHP collab

LQCD and the Proton Radius

CIPANP 2018, Palm Springs, CA
Using the data from the 8 ensembles, we perform the continuum-chiral extrapolations for the magnetic form factor.

$$G_E(Q^2) = F_1(Q^2) - \frac{Q^2}{4M_N^2} F_2(Q^2)$$

$$G_{Ev} = G_{Ep} - G_{En}$$

**Figure 3:** The 8 ensemble data for the normalized electric form factor.

**Electric Form Factor at the Physical Point**

- **$m_\pi = 139$ MeV $a=0.114$ fm**
  - [RBC/LHP, in prep.]
- **$m_\pi = 130$ MeV $a=0.094$ fm**
  - [1706.00469(ETMC)]
- **$m_\pi = 130..310$ MeV $a=0.06..0.12$ fm**
  - [1801.01635 (PNDME)]

**Key Points**
- The magnetic form factor includes terms up to $\ln(M/a)$.
- Contributions from disconnected fermion loops are included for the first time here.
- Disconnected fermion contributions since they cancel in the isospin limit, the isovector form factors do not include disconnected fermion contributions.
- The first column shows the terms included in the combination from the summation method.
- The second column shows the terms included in the fitting method.
- The notation is the same as that in Fig. 4. For the summation and two-state fit methods, the results will be reported in subsequent publications.
The chiral continuum extrapolation defined in Eq. (3) is obtained from the fit in Eq. (4), at $T = 8a$, however, there is an additional chiral log at the same order, i.e., pion cloud corrections. The results of the fits, with and without the respective FV correction, are much less stable since the point at $T = 0$.

As a result, the discrepancy could be due to suppressed pion cloud contributions since they cancel in the isospin limit, the individual proton and neutron form factors can be extracted. The isovector and isoscalar form factors do include disconnected fermion contributions, however, there is an additional chiral log term with the ansatz:

$$G_E(Q^2) = F_1(Q^2) - \frac{Q^2}{4M_N^2} F_2(Q^2)$$

$$G_{Ev} = G_{Ep} - G_{En}$$

Lattice results at physical $m_\pi$ overestimate $G_{Ev}(Q^2=0.4 \text{ GeV}^2)$ by 15-20%.

finite volume?

excited states?
"dipole" fits

\[
G(Q^2) \sim \frac{1}{(1 + Q^2/M_D^2)^2}
\]

\[
\langle r^2 \rangle = \frac{12}{M_D^2}
\]

model-independent fits (eg z-expansion)

\[
G(Q^2) \sim \sum a_k[z(t = -Q^2)]^k
\]

strong contributions from excited states

\[
\frac{r^2}{r^2 \text{ at } \pi^0} = \frac{1}{1 + Q^2/M_D^2}
\]

\[
\frac{r^2}{r^2 \text{ at } \pi^0} = \frac{12}{M_D^2}
\]

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\frac{r^2}{r^2 \text{ at } \pi^0} = \frac{1}{1 + Q^2/M_D^2}
\]

\[
\frac{r^2}{r^2 \text{ at } \pi^0} = \frac{12}{M_D^2}
\]
Smaller Momenta: Twisted Boundary Conditions

- Quantized lattice momenta for PBC
  \[ \psi(x + L) = \psi(L) \quad p_\mu = \frac{2\pi}{L_\mu} n_\mu \]
  with minimally acceptable \( m_\pi L \gtrsim 4 \)
  \( p_{\text{min}} \lesssim \frac{\pi}{2} m_\pi \approx 0.21 \text{GeV} \)
  \( Q_{\text{min}}^2 \approx 0.05 \text{GeV}^2 \)

- Twisted BC
  \[ \psi(x + \hat{L}_\mu) = e^{i\theta_\mu} \psi(x) \]
  arbitrary momenta \( p_\mu = \frac{2\pi}{L_\mu} n_\mu + \frac{\theta_\mu}{L_\mu} \)
  baryons with new twisted valence "flavor" \( r \)
  \[ \chi_\Sigma_r = \frac{1}{\sqrt{2}} ( [rud] + [rdu] ) , \]
  \[ \chi_\Lambda_r = \frac{1}{\sqrt{6}} ( 2[udr] - [rud] - [dru] ) , \]
  no sea twisted flavor \( \Rightarrow \) additional finite volume effects
  [F.J.Jiang, B.Tiburzi (0810.1495)]
Expansion in Lattice Momentum: Correlators

[de Divitiis, Petronzio, Tantalo (2012)]
compute correlator expansion

\[ C(q) = C(0) + \delta p_\mu \frac{\partial C(q)}{\partial q_\mu} + \frac{1}{2} \delta p_\mu \delta p_\nu \frac{\partial^2 C(q)}{\partial q_\mu \partial q_\nu} + \ldots \]

using propagator derivatives

\[ \frac{\delta \hat{D}^{-1}}{\delta p_\mu} = -\hat{D}^{-1} \frac{\delta \hat{D}}{\delta p_\mu} \hat{D}^{-1} \]
\[ \frac{\delta^2 \hat{D}^{-1}}{\delta p_\mu^2} = -\hat{D}^{-1} \frac{\delta^2 \hat{D}}{\delta p_\mu^2} \hat{D}^{-1} + 2\hat{D}^{-1} \left( \frac{\delta \hat{D}}{\delta p_\mu} \right) \hat{D}^{-1} \left( \frac{\delta \hat{D}}{\delta p_\mu} \right) \hat{D}^{-1} \]

"tadpole" (conserved) vector current

Implementation on a lattice: compute nucleon correlators with "sequential" propagators

\[ C^{(n)}_2 = \]
\[ C^{(n)}_3 = \]
Expansion in Lattice Momentum: Matrix Elements

Estimator for matrix elements

\[
R_N^X = \frac{C_3^{\delta_X^{q,\mu}}(\vec{p}, \vec{p}', \tau, T)}{\sqrt{C_2(\vec{p}, T)C_2(\vec{p}', T)}} \quad R_S = \sqrt{\frac{C_2(\vec{p}, T - \tau)C_2(\vec{p}', \tau)}{C_2(\vec{p}', T - \tau)C_2(\vec{p}, \tau)}} \\
R_X^{q,\mu}(\vec{p}, \vec{p}', \tau, T) = R_N^X R_S = M_X^{q,\mu}(\vec{p}, \vec{p}') + O(e^{-\Delta E_{10}(\vec{p})\tau}) + O(e^{-\Delta E_{10}(\vec{p}')}(T - \tau)) + O(e^{-\Delta E_{\text{min}}T})
\]

\[
\sum_{\lambda, \lambda'} \bar{u}(\vec{p}, \lambda) \Gamma_{\text{pol}} u(\vec{p}', \lambda') \langle p', \lambda' | \delta_X^{q,\mu} | p, \lambda \rangle \quad \text{converges to the ground state with } T \to \infty
\]

Vector current insertion

\[
R_V^0 = 1, \quad \partial_1 R_V^2 = -\frac{i}{2m} G_M(0), \quad \partial_1^2 R_V^0 = -\frac{1}{4m^2} - \frac{1}{3} r_E^2, \quad \mu = 2im (R_V^2)', \quad r_E^2 = -\frac{3}{4m^2} - 3 \frac{(R_V^0)''}{R_V^0}
\]
Isovector Electric Form Factor

\[ G_E(Q^2) = F_1(Q^2) - \frac{Q^2}{4M_N^2} F_2(Q^2) \]
\[ G_{Ev} = G_{Ep} - G_{En} \]

comparison of \( G_{Ev} \) slope at \( Q_2 = 0 \)
- z-expansion fit vs.
- twist-derivative method

\[ m_\pi = 135 \text{ MeV} \quad a = 0.093 \text{ fm} \quad 64^4 \text{ (BMWC)} \]

[N. Hasan et al, PRD97: 034504 (1711.11385)]
"dipole" fits

\[ G(Q^2) \sim \frac{1}{(1 + Q^2/M_D^2)^2} \]

\[ \langle r^2 \rangle = \frac{12}{M_D^2} \]

model-independent fits (eg z-expansion)

\[ z = \frac{\sqrt{t_{\text{cut}} - t - \sqrt{t_{\text{cut}} - t_0}}}{\sqrt{t_{\text{cut}} - t + \sqrt{t_{\text{cut}} - t_0}}} \]

\[ G(Q^2) \sim \sum a_k [z(t = -Q^2)]^k \]

strong contributions for excited states

- [LHPC 2017], deriv
- [RBC/LHP, in prep.], z-expn
- [LHPC 2017], z-expn
- [ETMC 2017], dipole(o), z-expn(f)
- [NME 2018], dipole(o), z-expn(f)
Derivatives wrt. Initial and Final Momenta

Evaluate the radius from varying the initial and final momenta
[B.Tiburzi, 1407.1459] \[ \frac{\partial^2}{\partial p'_i \partial p_i} \langle N(p') J^0 N(p) \rangle \]

No tadpole insertions in propagator derivatives

\[ q\mu = \delta p'_\mu - \delta p_\mu \]

\[ m_\pi = 135 \text{ MeV} \quad a = 0.116 \text{ fm} \quad 48^4 \text{ (BMWc)} \]

[N.Hasan, LATTICE2017]

\[ \frac{\partial}{\partial \vec{p}'} \frac{\partial}{\partial \vec{p}} C^0_3(\vec{p}', \vec{p}, T, \tau)|_{\vec{p} = \vec{p}' = 0} \]
\[ = \frac{1}{4m^2} [F_1 + 2F_2] + \frac{1}{3} F_1[r_1]^2 \]

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**Graph:**

- **PRELIMINARY**
- **CODATA**
- **\( \mu \)H**

**Z-expansion fit**

**Mixed derivatives w.r.t \( \vec{p}, \vec{p}' \)**

**Second derivative w.r.t \( \vec{p} \)**
Sachs magnetic form factor

\[
\langle p' | V_\mu | p \rangle = \bar{u}' \left[ F_1 \gamma_\mu + F_2 \frac{\sigma_{\mu\nu} q^{\nu}}{2m_N} \right] u
\]

\[
G_M = F_1(Q^2) + F_2(Q^2)
\]

\[
\partial_1 R_V^2 = -\frac{i}{2m} G_M(0)
\]

Induced pseudoscalar form factors

\[
\langle p' | A_\mu | p \rangle = \bar{u}' \left[ G_A \gamma_\mu \gamma_5 + G_P \frac{q_\mu \gamma_5}{m_N} \right] u
\]

\[
G_P(0) = m^2 \left( \partial_1^2 R_A^3 + \partial_2^2 R_A^3 - 2 \partial_3^2 R_A^3 \right)
\]

\[m_\pi = 135 \text{ MeV} \quad a=0.093 \text{ fm} \quad 64^4 \text{ (BMWc)} \]

[N.Hasan et al, PRD97: 034504 (1711.11385)]
Summary

- Multiple lattice results for nucleon form factors at the physical point
  \textit{Chiral extrapolation in }m_\pi\text{ no longer required}

- Large systematic bias seen by all lattice groups
  \textit{Overestimate }G_{Ev}(Q^2=0.4 \text{ GeV}^2)\text{ by }15\text{-}20\%
  \textit{Underestimate isovector radius by }20\text{-}25\%

- Precision for charge radius is insufficient for any conclusions
  \textit{Both statistical and systematic uncertainty}

- Multiple potential sources of systematic uncertainty to explore
  \textit{Excited state effects}
  \textit{Finite volume effects}
  \textit{Zero-momentum extrapolation}

- New promising methods for nucleon structure at zero momentum
  \textit{Charge radius}
  \textit{Form factors vanishing from forward matrix elements}