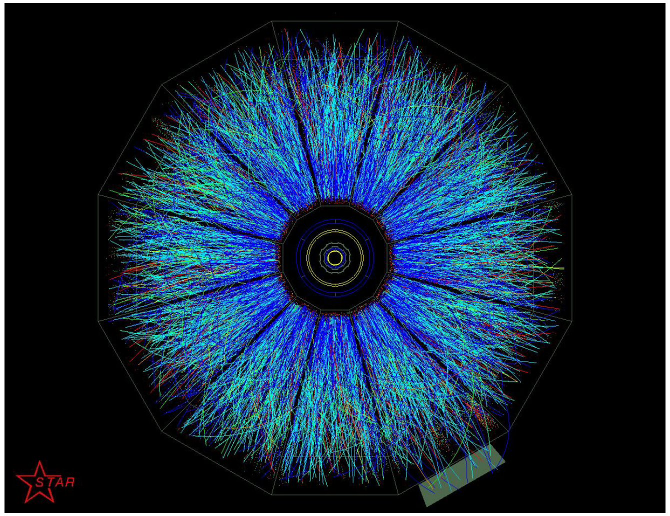
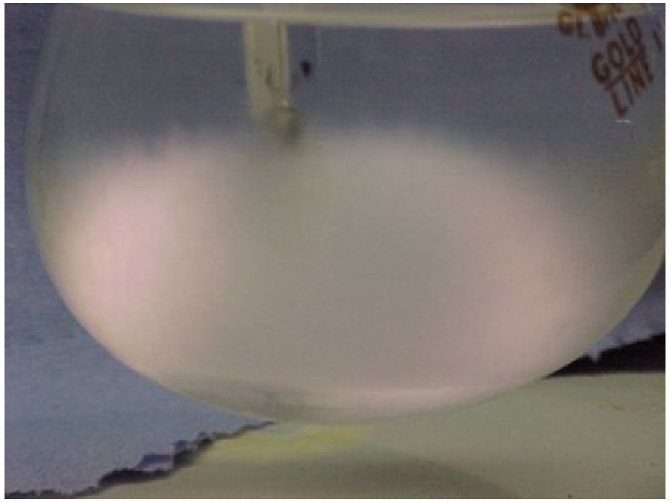
Simulating stochastic fluids

Thomas Schäfer

North Carolina State University





With C. Chattopadhyay, J. Ott, V. Skokov

References: 2302.00720, 2304.07279, 2403.10608.

Dynamical Theory

What is the dynamical theory near the critical point?

The basic logic of fluid dynamics still applies. Important modifications:

- Critical equation of state.
- Stochastic fluxes, fluctuation-dissipation relations.
- Possible Goldstone modes (chiral field in QCD)

Chiral phase transition: Model G (Rajagopal & Wilzcek)

Possible critical endpoint: Model H (Son & Stephanov)

Digression: Diffusion

Consider a Brownian particle

$$\dot{p}(t) \;= -\gamma_D p(t) + \zeta(t) \qquad\qquad \langle \zeta(t)\zeta(t')\rangle = \kappa\delta(t-t')$$

drag (dissipation) white noise (fluctuations)

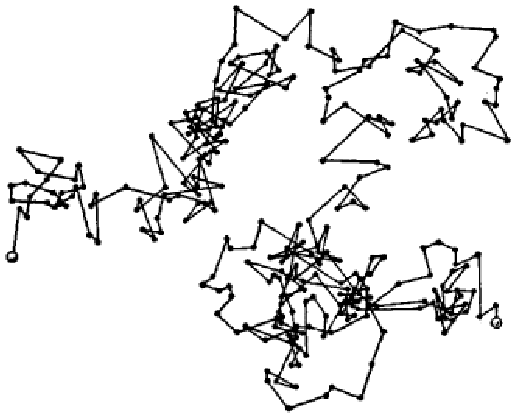
For the particle to eventually thermalize

$$\langle p^2 \rangle = 2mT$$

drag and noise must be related

$$\kappa = \frac{mT}{\gamma_D}$$

Einstein (Fluctuation-Dissipation)



Hydrodynamic equation for critical mode

Equation of motion for critical mode ϕ ("model H")

$$\frac{\partial\phi}{\partial t} = \kappa\,\nabla^2\frac{\delta\mathcal{F}}{\delta\phi} - g\left(\vec{\nabla}\phi\right)\cdot\frac{\delta\mathcal{F}}{\delta\vec{\pi}^T} + \zeta \qquad (g=1)$$

Diffusion Advection Noise

Equation of motion for momentum density π

$$\frac{\partial\vec{\pi}^T}{\partial t} = \eta\,\nabla^2\frac{\delta\mathcal{F}}{\delta\vec{\pi}^T} + g\left(\vec{\nabla}\phi\right)\cdot\frac{\delta\mathcal{F}}{\delta\phi} - g\left(\frac{\delta\mathcal{F}}{\delta\vec{\pi}^T}\cdot\vec{\nabla}\right)\vec{\pi}^T + \vec{\xi}$$

Free energy functional: Order parameter ϕ, momentum density $\vec{\pi} = w\vec{v}$

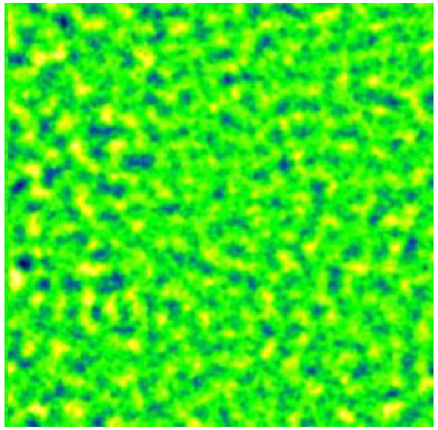
$$\mathcal{F} = \int d^3x\left[\frac{1}{2w}\vec{\pi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{m^2}{2}\phi^2 + \lambda\phi^4\right] \qquad D = m^2\kappa$$

Fluctuation-Dissipation relation

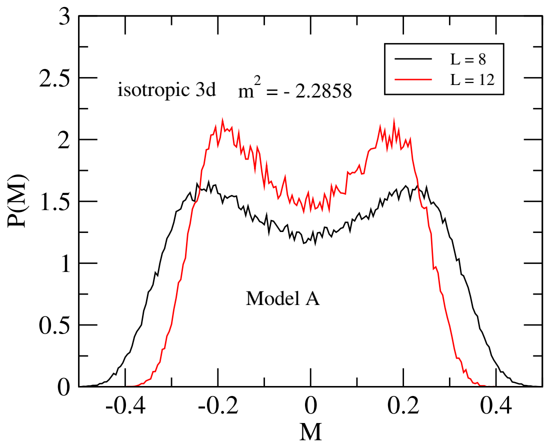
$$\langle\zeta(x,t)\zeta(x',t')\rangle = -2\kappa T\nabla^2\delta(x-x')\delta(t-t')$$

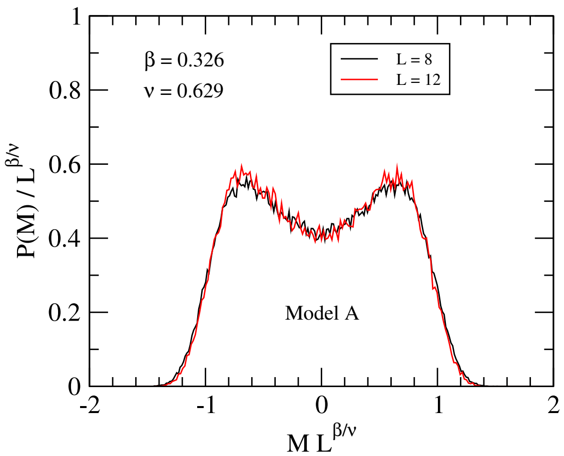
$$\langle\xi_i(x,t)\xi_j(x',t')\rangle = -2\eta T\nabla^2 P_{ij}^T\delta(x-x')\delta(t-t')$$

ensures $P[\phi,\vec{\pi}] \sim \exp(-\mathcal{F}[\phi,\vec{\pi}]/T)$



Tune m^2 to critical point $m^2 = m_c^2$ (Ising critical point)





Numerical realization

Stochastic relaxation equation ("model A")

$$\partial_t \psi = -\Gamma \frac{\delta \mathcal{F}}{\delta \psi} + \zeta \qquad \langle \zeta(x,t)\zeta(x',t')\rangle = \Gamma T \delta(x-x')\delta(t-t')$$

Naive discretization

$$\psi(t+\Delta t) = \psi(t) + (\Delta t)\left[-\Gamma \frac{\delta \mathcal{F}}{\delta \psi} + \sqrt{\frac{\Gamma T}{(\Delta t) a^3}}\,\theta\right] \qquad \langle \theta^2 \rangle = 1$$

Noise dominates as $\Delta t \to 0$, leads to discretization ambiguities in the equilibrium distribution.

Idea: Use Metropolis update

$$\psi(t+\Delta t) = \psi(t) + \sqrt{2\Gamma(\Delta t)}\,\theta \qquad p = min(1, e^{-\beta \Delta \mathcal{F}})$$

Numerical realization

Central observation

$$
\begin{aligned}
\langle \psi(t+\Delta t,\vec{x}) - \psi(t,\vec{x})\rangle &= -(\Delta t)\,\Gamma\,\frac{\delta \mathcal{F}}{\delta\psi} + O\left((\Delta t)^2\right) \\
\langle [\psi(t+\Delta t,\vec{x}) - \psi(t,\vec{x})]^2\rangle &= 2(\Delta t)\,\Gamma\, T + O\left((\Delta t)^2\right) .
\end{aligned}
$$

Metropolis realizes both diffusive and stochastic step. Also

$$P[\psi] \sim \exp(-\beta\mathcal{F}[\psi])$$

Note: Still have short distance noise; need to adjust bare parameters such as Γ, m^2, λ to reproduce physical quantities.

Numerical realization: Model H

Model H: Conserving update

$$\pi_\nu^{trial}(\vec{x}, t+\Delta t) = \pi_\nu(\vec{x}, t) + r_{\nu\mu}\,,$$
$$\pi_\nu^{trial}(\vec{x}+\hat{\mu}, t+\Delta t) = \pi_\nu(\vec{x}+\hat{\mu}, t) - r_{\nu\mu}\,,$$

$$r_{\nu\mu} = \sqrt{2\eta T(\Delta t)}\,\zeta_\nu\,.$$

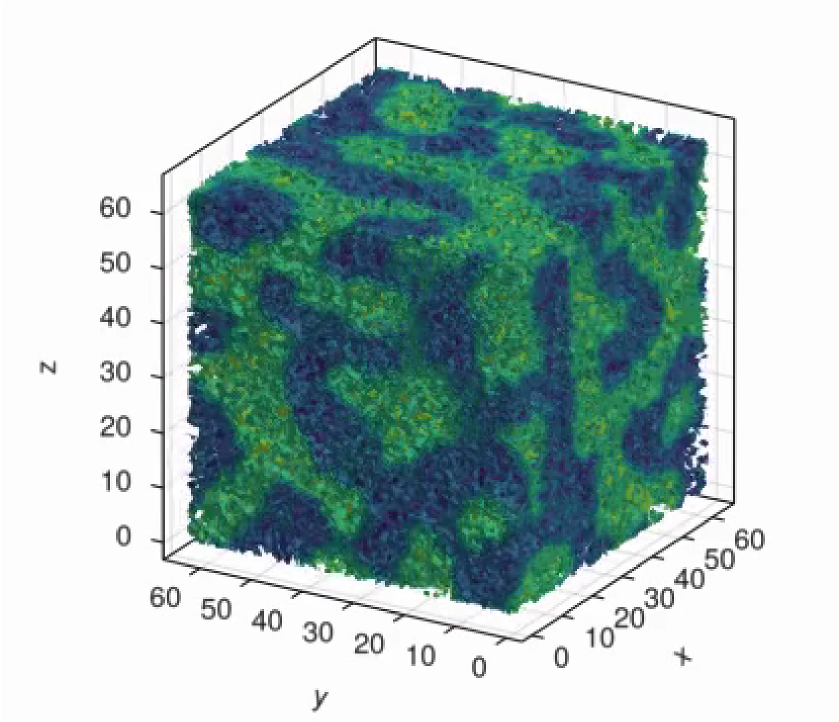
Advection (PB terms) conserves $\mathcal{H}$. On the lattice use "skew" discretized derivatives

$$\dot{\phi} = -\frac{1}{\rho}\,\pi_\mu^T\,\nabla_\mu^c \phi\,,$$
$$\dot{\pi}_\mu^T = -\left[\frac{1}{2}\nabla_\nu^c\left(\frac{1}{\rho}\pi_\nu^T\pi_\mu^T\right) + \frac{1}{2\rho}\pi_\nu^T\,\nabla_\nu^c\pi_\mu^T + \left(\nabla_\mu^c\phi\right)\left(\nabla_\nu^c\nabla_\nu^c\phi\right)\right]\,,$$

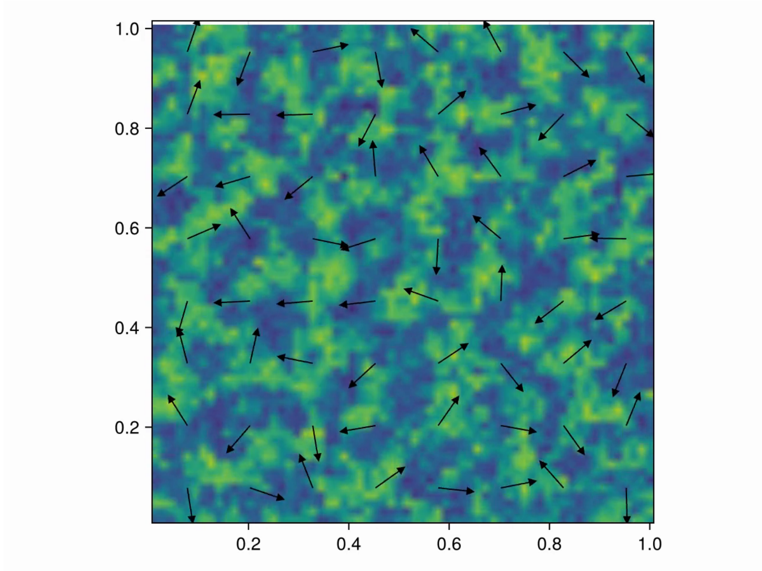
and project on π_μ^T using Fourier transforms.

Numerical results (critical Navier-Stokes)

Order parameter (3d)



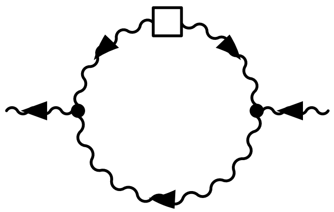
Order parameter/velocity field (2d)



Renormalized viscosity

Renormalization of η

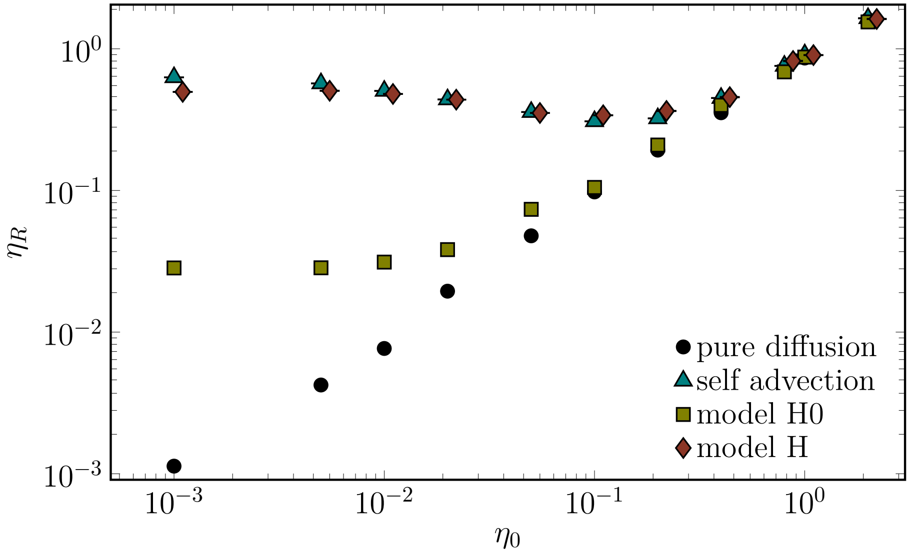
"Stickiness of shear waves"



$$\eta_R = \eta + \frac{7}{60\pi^2}\frac{\rho T \Lambda}{\eta}$$

Leads to minimum viscosity

Top: Model H

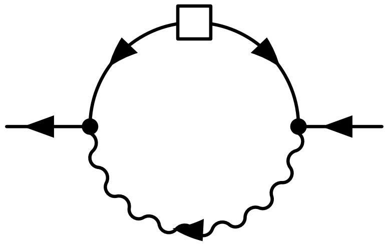


Middle: No self-advection

Bottom: No advection

Relaxation Rate

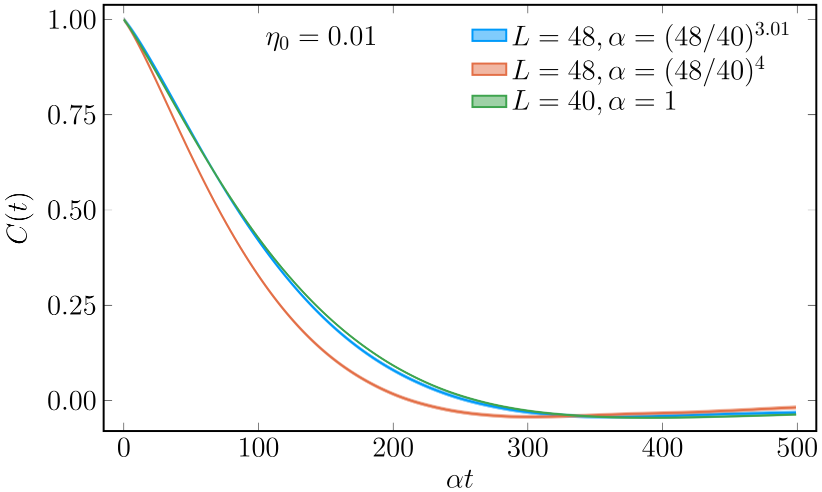
Order parameter relaxation rate



$$\Gamma_k = \frac{\Gamma}{\xi^4} (k\xi)^2 \left(1 + (k\xi)^2\right) + \frac{T}{6\pi\eta_R \xi^3} K(k\xi)$$

Crossover from $\tau_R \sim \xi^4$ at large η_R to $\tau_R \sim \xi^3$ for small η_R

$$C(t) = \langle \phi_k(0)\phi_{-k}(t) \rangle$$

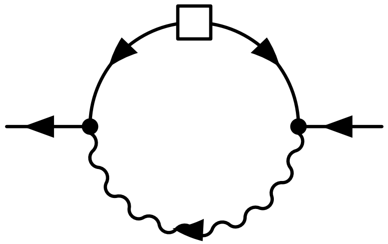


Dynamic Scaling:

$$z(\eta = 0.01) = 3.07$$

Relaxation Rate

Order parameter relaxation rate



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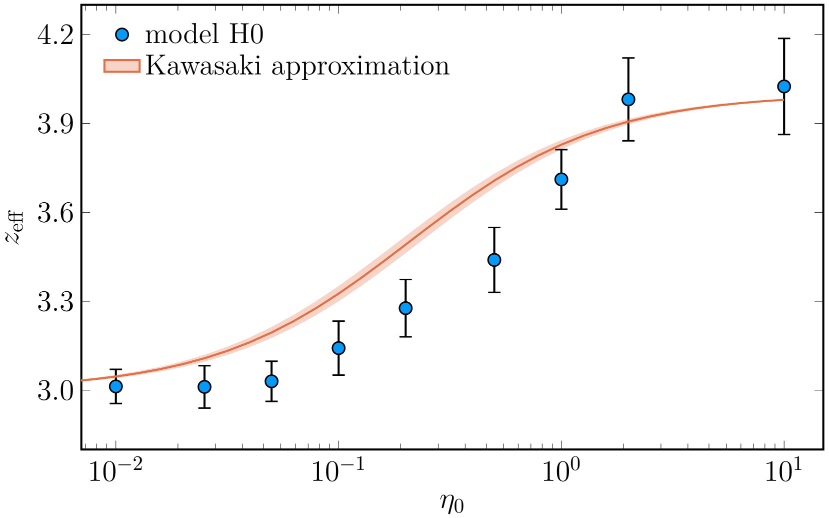
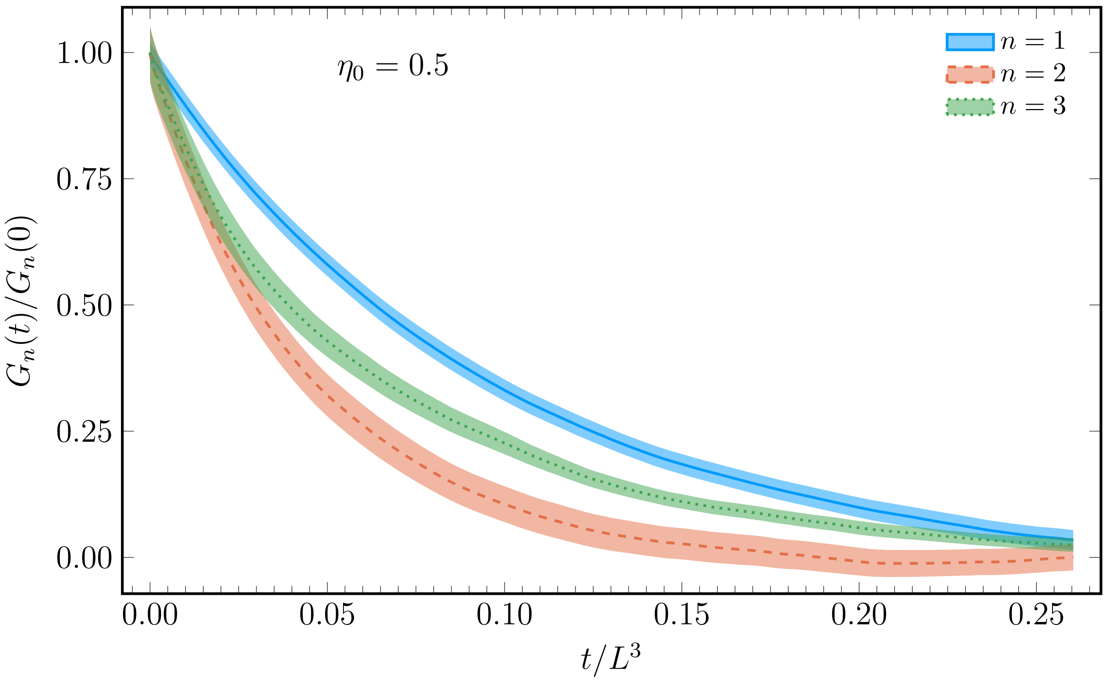


Crossover from $\tau_R \sim \xi^4$ at large η_R

to $\tau_R \sim \xi^3$ for small η_R

Evolution of higher moments



$$G_n(t) = \langle M^n(t) M^n(0) \rangle \,, \qquad M(t) = \int_V d^3x \, \phi(\vec{x}, t)$$

Summary and Outlook

Numerical simulation of stochastic fluid dynamics, observed renormalization of shear viscosity and dynamical scaling.

Outlook (post BESII reveal): 1) We still want to predict the impact of a possible CEP on observables. 2) There is a crossover transition with non-trivial chiral and baryon number susceptibilities. Can we detect that in the data? 3) What is the impact of fluctuations on small systems?

For this purpose, we need to extend the present framework to full (relativistic) fluid dynamics, or couple the simulations to fixed relativistic background flow (no backreaction).