Non-Gaussian Correlations in Hydrodynamics with *Velocity* Fluctuations

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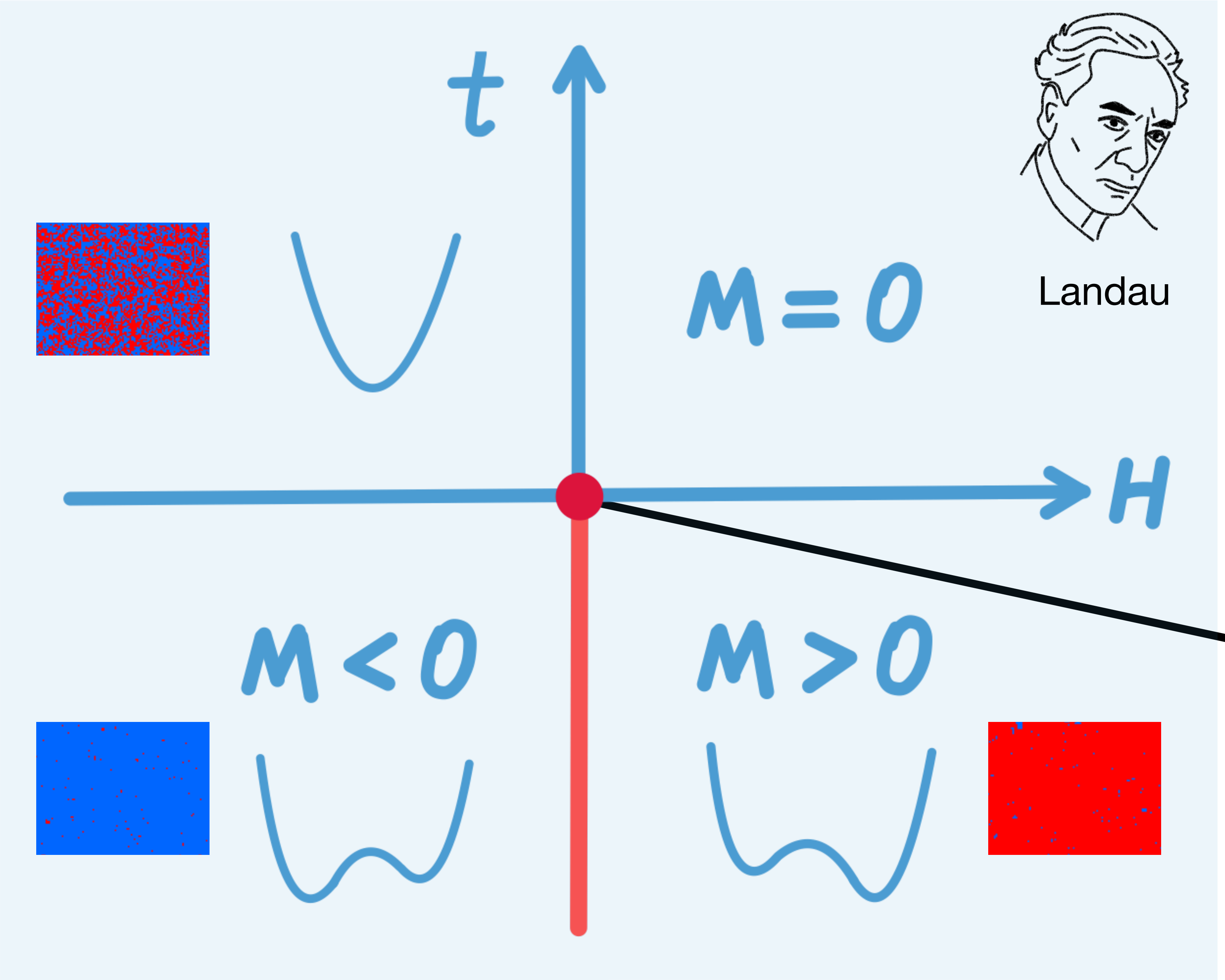
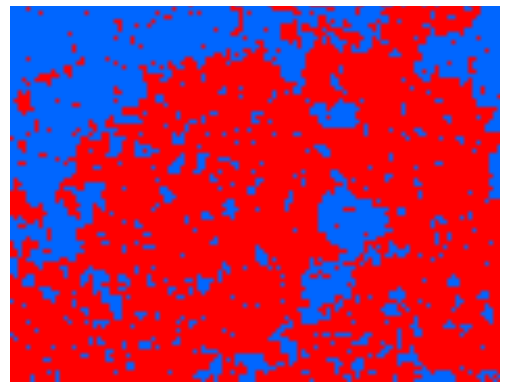
NCBJ

Thermodynamic fluctuations

- Fluctuations are important measures of criticality, due to the reduction of the number of effective DOFs near the critical point.

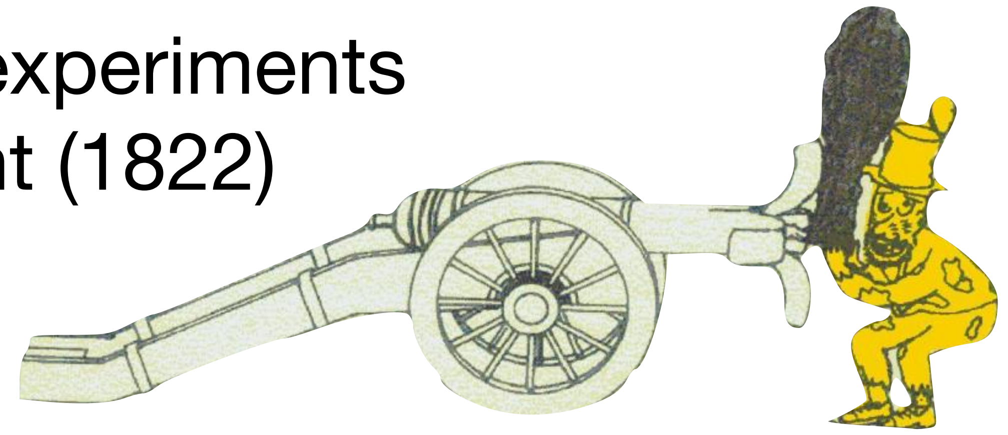


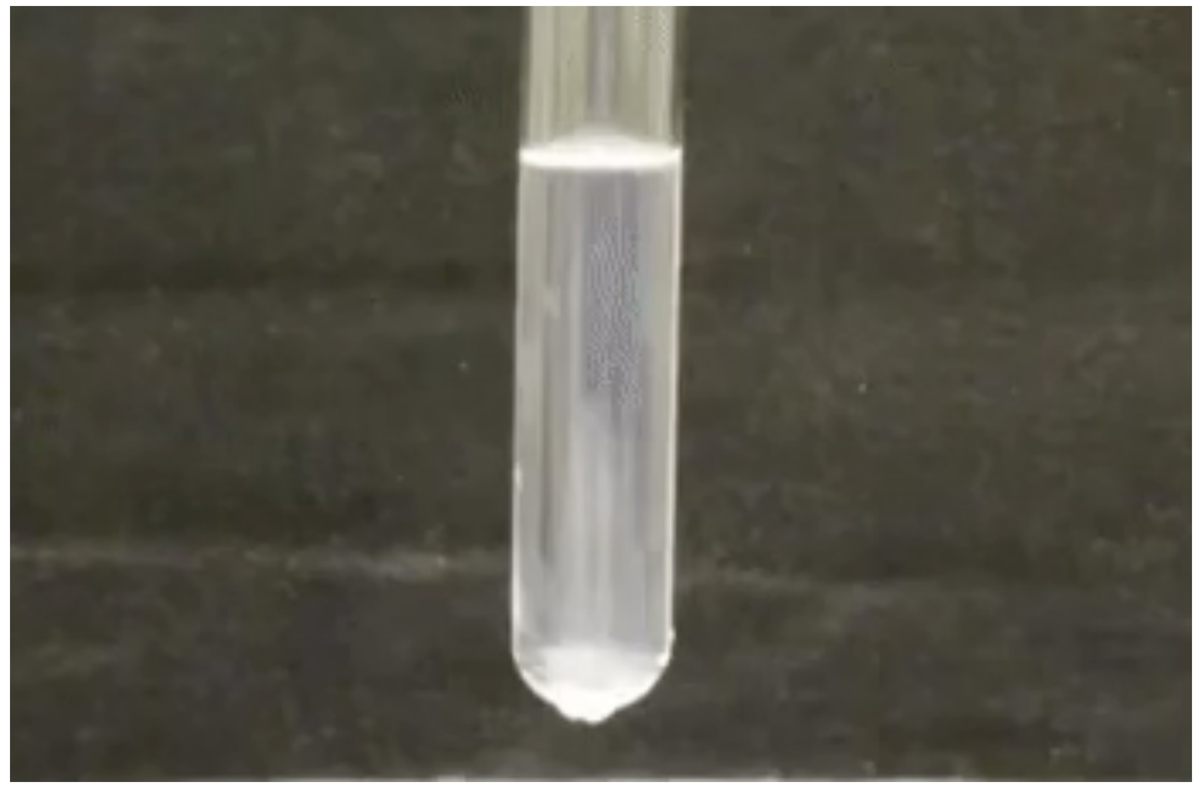
Ising phase diagram



$\xi \to \infty$

Charles Cagniard's gun barrel experiments for the discovery of critical point (1822)

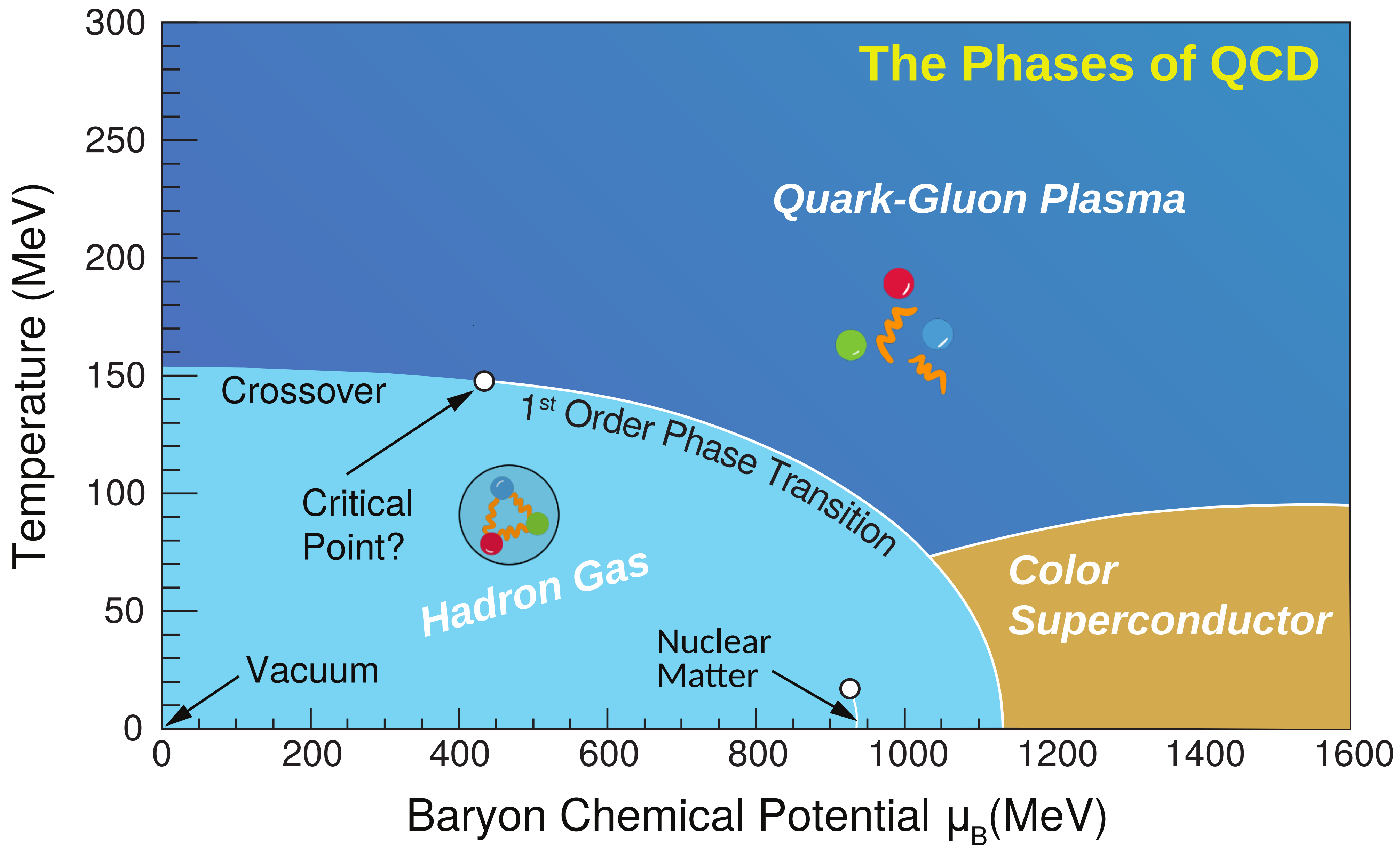


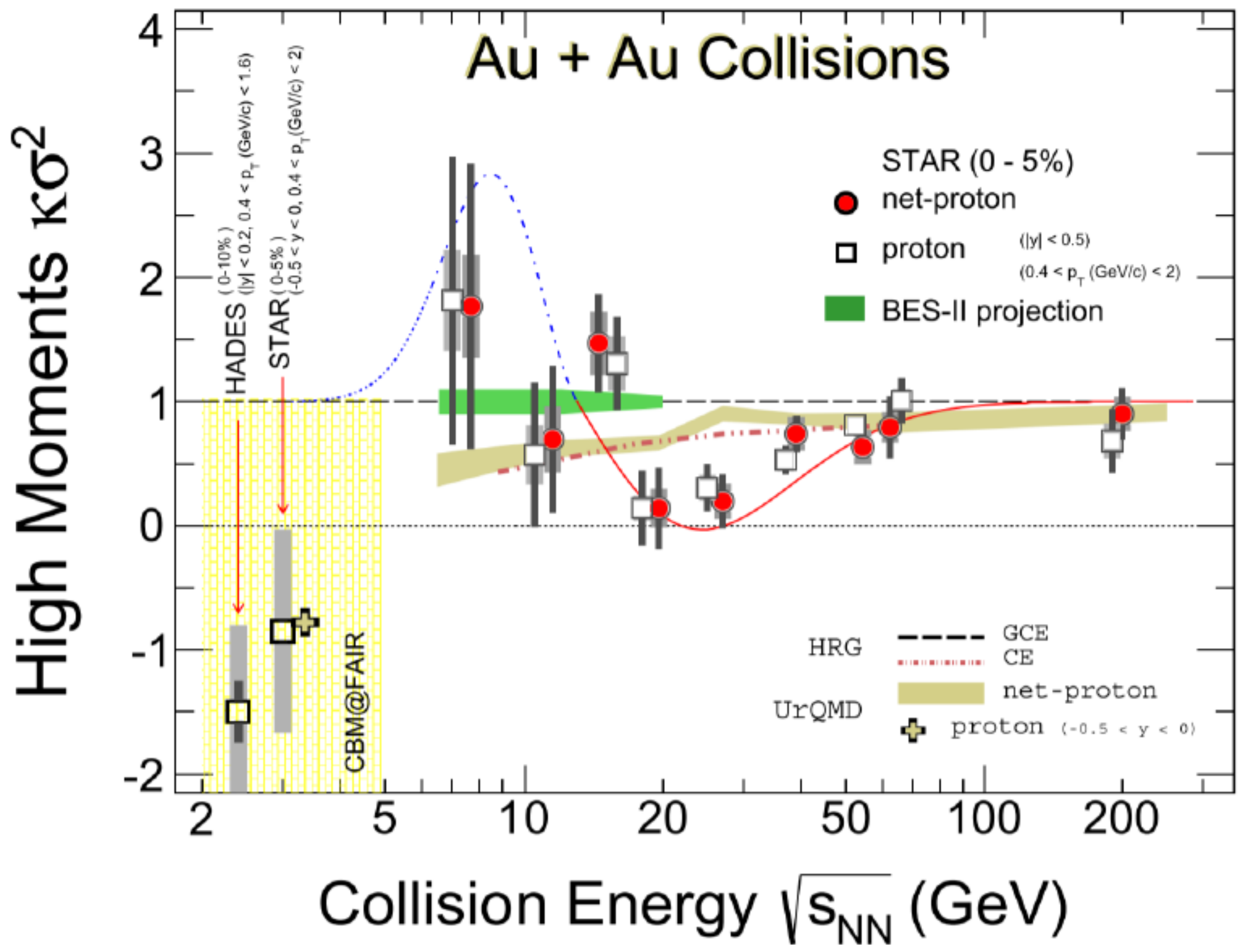


Critical opalescence: $\xi \leftrightarrow \lambda_{\text{light}}$

Thermodynamic fluctuations and QCD critical point

- The intriguing hint of the QCD critical point from BES-I data relies on the assumption of **equilibrated** and **static** medium. Stephanov, 1104.1627; STAR, 2112.00240



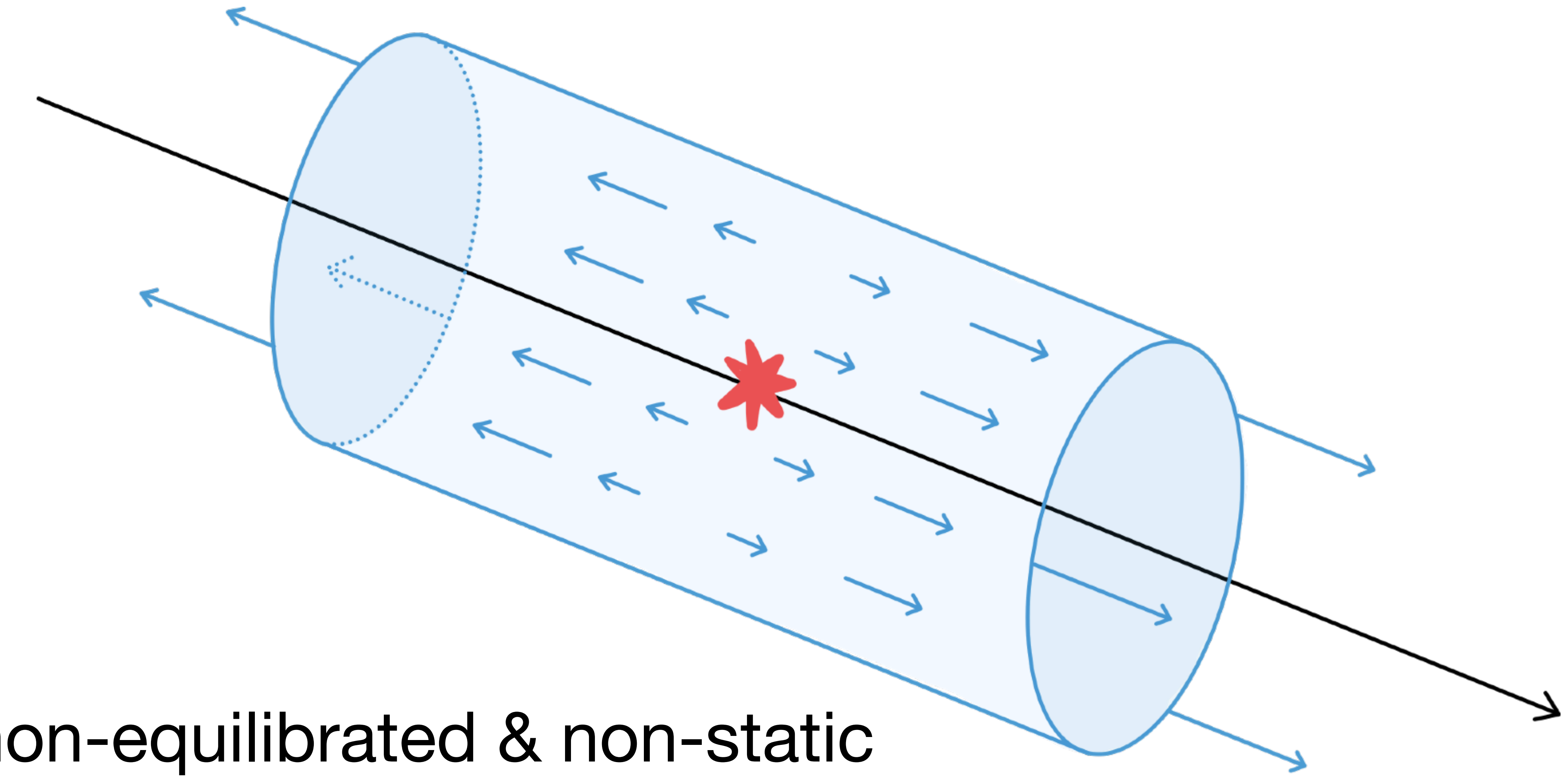


static universality class of 3D Ising model, see also M. Pradeep's talk

- Recent progress: **non-equilibrium** thermodynamic fluctuations, in particular, non-Gaussian density fluctuations dynamics with **non-fluctuating moving** background near the critical point. E.g., XA et al, 2212.14029 and many others

Velocity and its fluctuations in heavy-ion collisions

- QGP matter produced in HIC exhibits event-by-event collective motion.



non-equilibrated & non-static

Velocity fluctuations are essential in measuring flow harmonics v_n and quantifying non-flow contribution.

XA et al, 2312.17237

Lagrangian specification of the flow: observer (Ponyo) in the LRF of a fluid cell (fish)

Ponyo (2008)

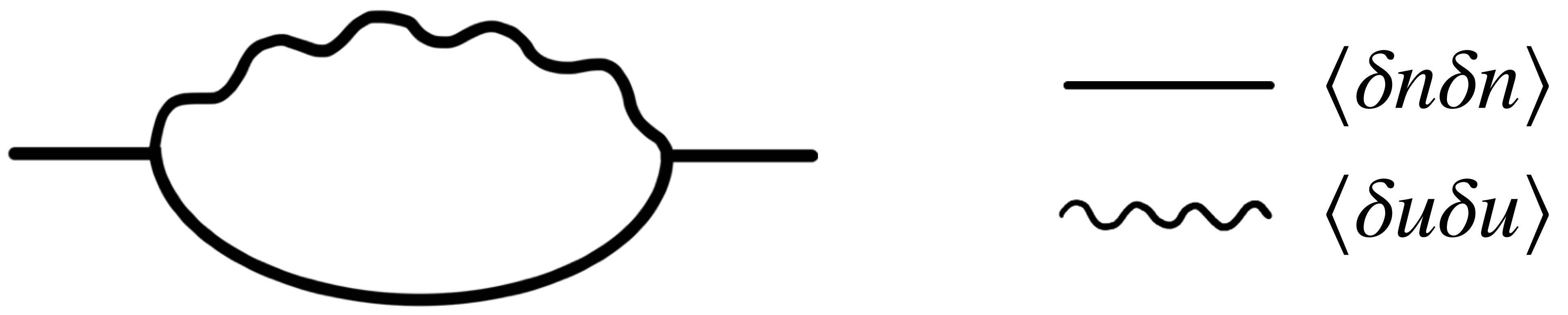
Velocity fluctuations and QCD critical point

- The dynamic universality class of a critical point depends on the **relevant hydro DOFs.** Critical points in the same static universality class (e.g., 3D Ising) could differ in dynamic universality class by the **involvement of velocity fluctuations** (e.g., Model B vs Model H).

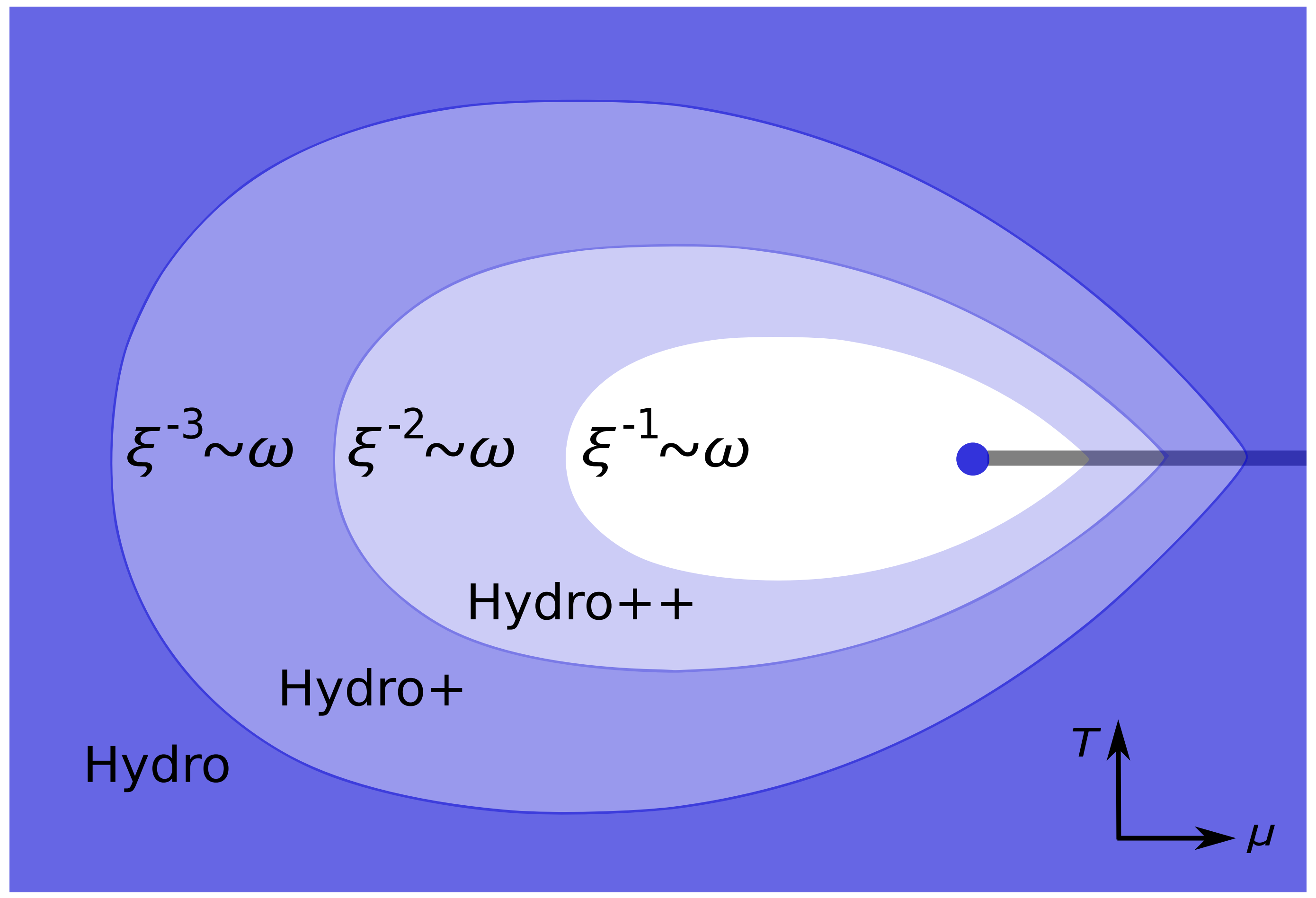
Model B
baryon density

moving medium → advection

Model H
baryon density couples to energy-momentum tensor (nonlinear mode coupling)



- Velocity fluctuations matter near the critical point due to time scale hierarchy (Hydro++, Model H), and are equally important as other thermodynamic fluctuations far from it (hydro-kinetic theory).



Velocity fluctuations in Brownian motion

- Einstein's formula for diffusion coefficient in $\partial_t \rho = D \nabla^2 \rho$: Einstein, 1905

$$D = \lim_{t \to \infty} \frac{1}{2t} \langle \Delta x^2(t) \rangle = \int_0^\infty d\tau \langle v(\tau) v(0) \rangle$$

Kubo formula

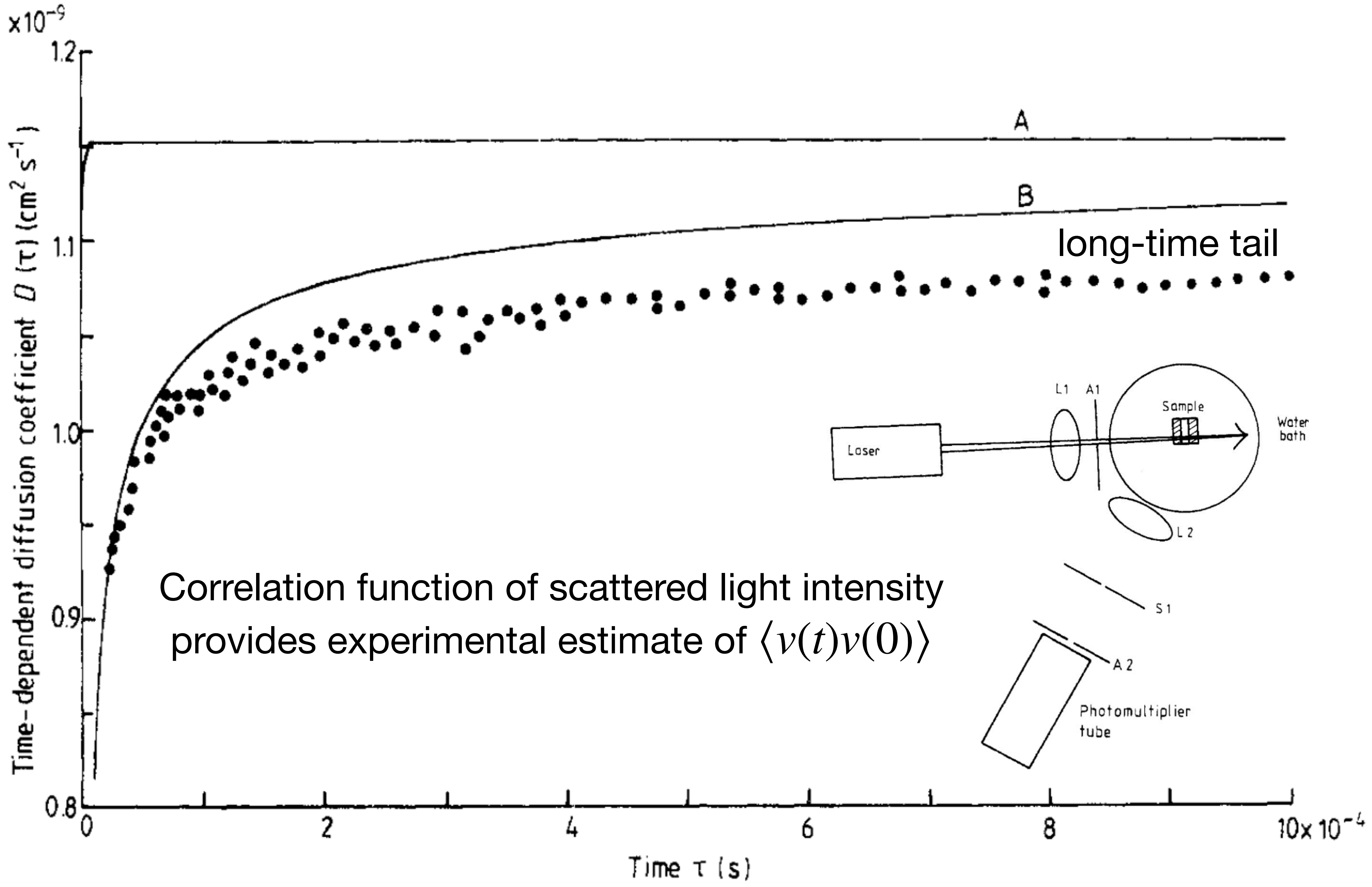
- Measurable long-time behavior:

$$\langle v(t) v(0) \rangle \sim e^{-\mu t} \quad \rightarrow \quad D \sim \mu^{-1}$$

with only dissipation

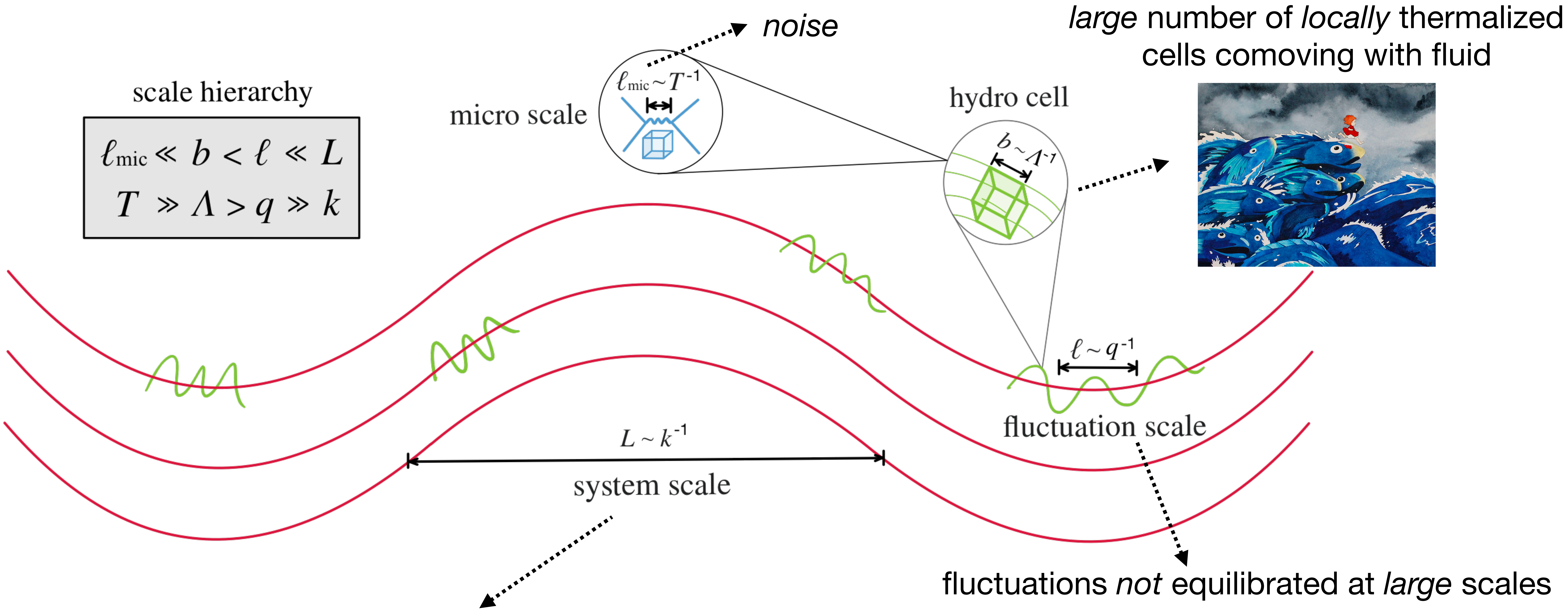
$$\langle v(t) v(0) \rangle \sim t^{-3/2} \quad \rightarrow \quad D \sim t^{-1/2}$$

with also fluctuation



Hydrodynamic scales

- The scale hierarchy ensures: 1) local thermalization; 2) small Knudsen number; 3) separation of fluctuations and background.



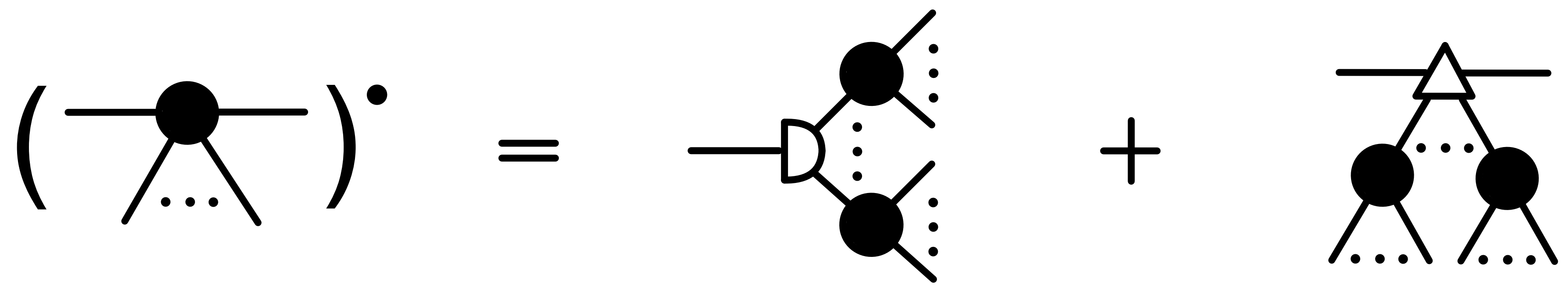
evolution described by a set of conservation equations $\partial_t \psi = \nabla \cdot (\text{flux}\,[\psi])$ where ψ = conserved quantities

Correlator evolution equations

- The deterministic approach provides the truncated evolution equations for the fluctuation correlators. XA et al, 2009.10742, 2212.14029

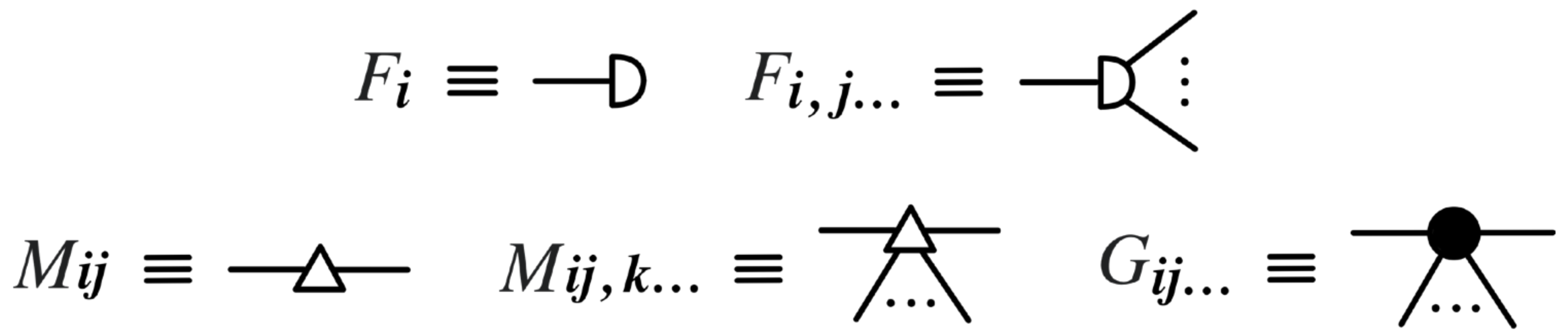
$$\partial_t G_n = \mathscr{F}\left[\langle\psi\rangle, G_2, G_3, \ldots, G_n\right] + \mathcal{O}(\varepsilon^n) \qquad \text{where} \qquad G_n \sim \varepsilon^{n-1}, \qquad F_i \sim 1, \qquad M_{ij} \sim \varepsilon\,.$$

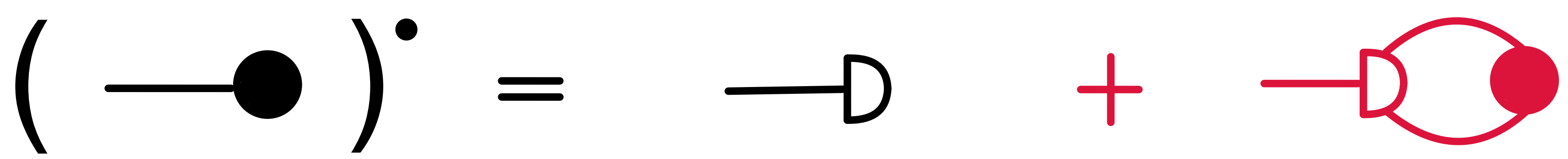
loop-expansion parameters: $\varepsilon \sim (\xi/\ell)^3 \sim 1/$number of correlated volumes $\quad \phi \sim \sqrt{\epsilon}$ CLT!



drift noise

all *combinatorial* configurations of trees





1-pt equation including leading loop

conventional hydro equations *one loop (renormalization & long-time tails)*

see T. Schaefer's talk for alternative, stochastic approach using Model H

Relativistic dynamics

- **1-pt**: covariant Langevin equations

$$\breve{u}\cdot\partial\breve{\psi}_i = F_i(\breve{\psi}) + \xi_i(\breve{\psi})$$

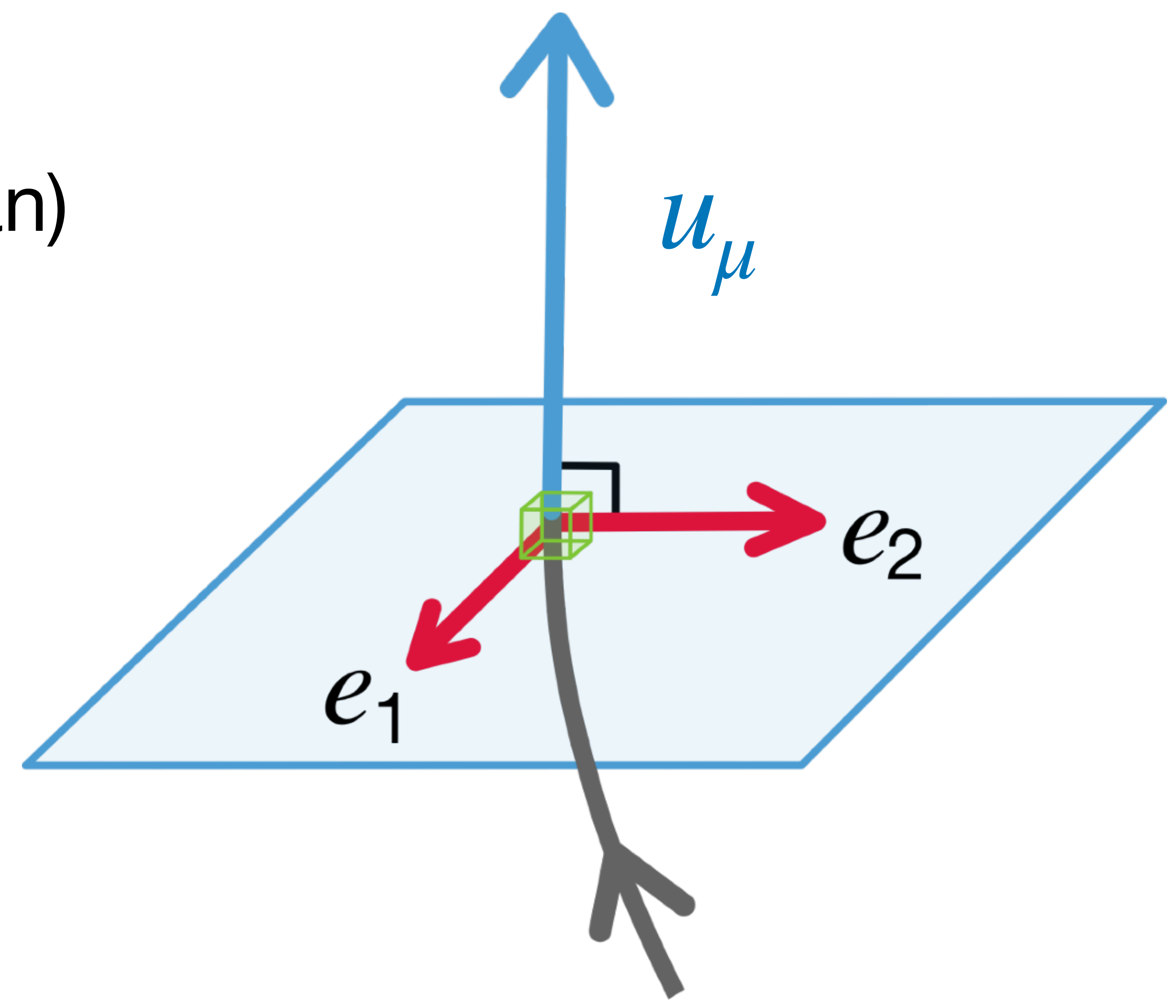
e.g., for velocity field $\psi = u_\mu$,

$$F_\mu = -\frac{1}{w}\Delta_\mu^{\ \lambda}\left(\partial_\lambda p + \partial^\alpha(2\eta\Delta_{\lambda\alpha\nu\beta} + \zeta\Delta_{\lambda\alpha}\Delta_{\nu\beta})\partial^\beta u^\nu\right) \qquad \xi_\mu = -\frac{1}{w}\Delta_\mu^{\ \lambda}\partial^\nu H^{ab}_{\lambda\nu}\eta_{ab}$$

where $H^{\mu\nu}_{ab} = \sqrt{\frac{T\eta}{2}}\left(e^\mu_a e^\nu_b + e^\mu_b e^\nu_a - \frac{2}{3}\Delta^{\mu\nu}\delta_{ab}\right) + \sqrt{\frac{T\zeta}{3}}\Delta^{\mu\nu}\delta_{ab}$ (FDT)

$$\langle\eta_{ab}(x_1)\,\eta_{cd}(x_2)\rangle = (\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})\,\delta^{(4)}(x_1 - x_2) \quad \text{(Gaussian)}$$

where we introduced *spatial triad* e^a_μ $(a = 1,2,3)$ perp. to temporal vector u_μ.



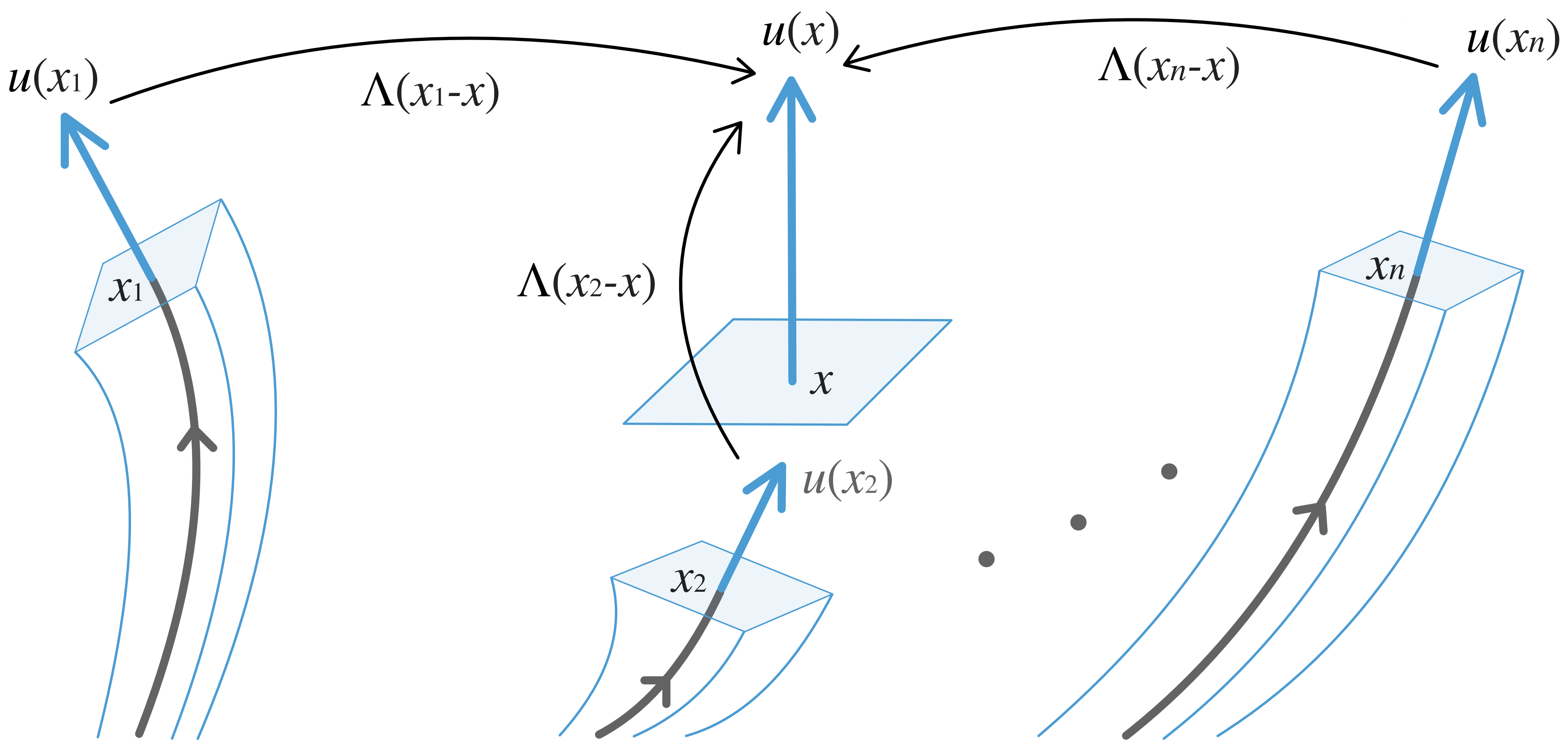
- **n-pt**: separation of the evolution at midpoint and the relative motion to it in the *equal-time* hypersurface.

Confluent formulation: correlator and derivative

- Confluent formulation: covariant description for the comoving fluctuations.

See XA et al, 2212.14029 for more details

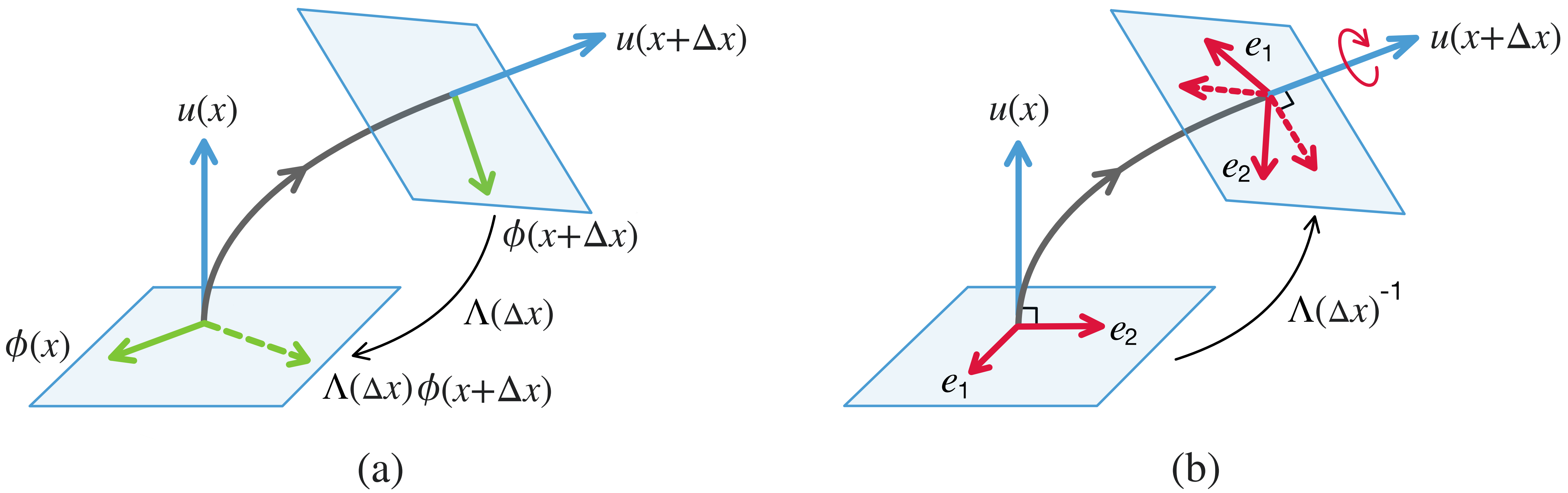
Confluent correlator $\bar{G}$



$$\bar{G}_{i_1\ldots i_n} = \Lambda_{i_1}^{\ j_1}(x-x_1)\ldots\Lambda_{i_n}^{\ j_n}(x-x_n)\bar{G}_{j_1\ldots j_n}$$

boost all fields (measured at their own local rest frame) to one common frame (chosen at their midpoint)

Confluent derivative $\bar{\nabla}$



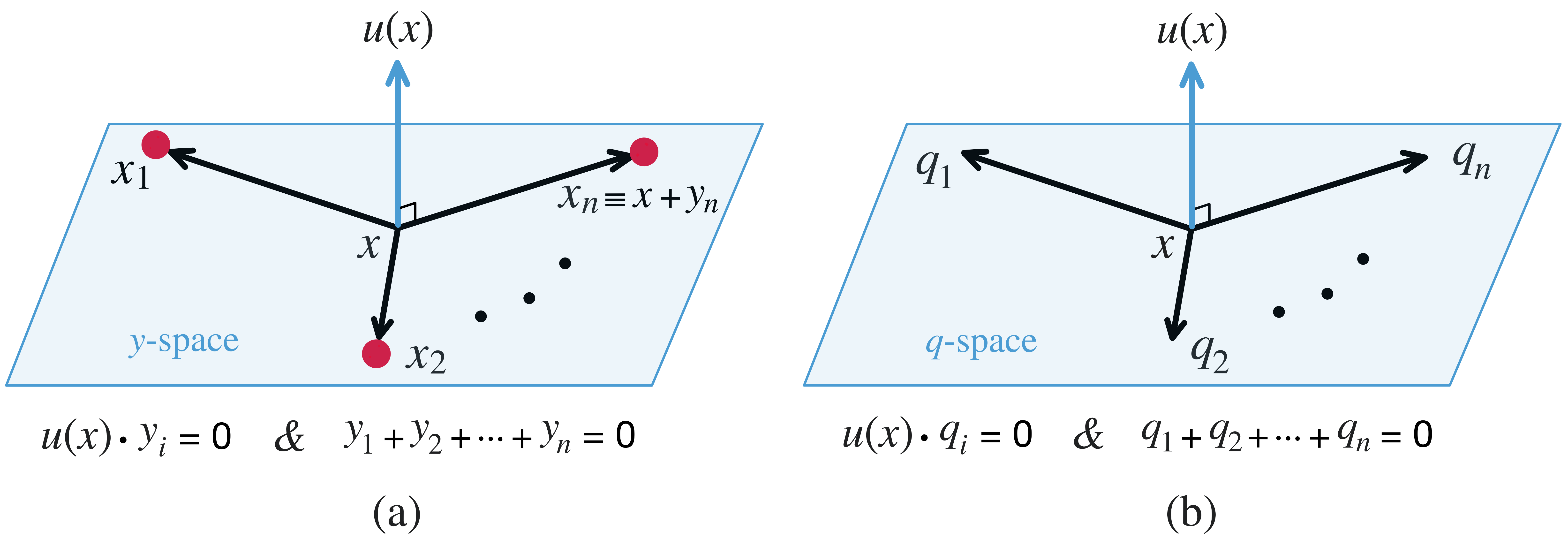
$$\bar{\nabla}_\mu \bar{G}_{i_1\ldots i_n} = \partial_\mu \bar{G}_{i_1\ldots i_n} - n\left(\bar{\omega}^{j_1}_{\mu i_1}\bar{G}_{j_1\ldots i_n} + \mathring{\omega}^{a}_{\mu b}\, y_1^b \partial_a^{(y_1)}\bar{G}_{i_1\ldots i_n}\right)_{\text{perm.}}$$

compare the difference of a given field along the time direction in one frame, with the equal-time constraint preserved

Confluent formulation: Wigner function

- Equal-time **constraint** $u \cdot y = 0$ allows us to write $y^\mu = y^a e_a^\mu$. Thus we only need confluent n-pt Wigner transform between y^a and q^a. XA et al, 2212.14029

$$W_n(x; q_1^a, \ldots, q_n^a) = \int \prod_{i=1}^{n} \left(d^3 y_i^a \, e^{-i q_{ia} y_i^a} \right) \delta^{(3)} \left(\frac{1}{n} \sum_{i=1}^{n} y_i^a \right) \bar{G}_n(x + e_a y_1^a, \ldots, x + e_a y_n^a)$$



"While the bottom-up approach is useful in order to calculate two-point correlation functions, it is not immediately obvious how it should be generalized for the calculation of n-point correlation functions." Romatschke, 2019

Stochastic variables with constraint

- Velocity is not a suitable primary variable due to the **constraint** $u^2 = -1$, i.e.,

$$\langle \breve{u}_\mu \rangle \neq u_\mu \,.$$

Thus we introduce a fluctuating 3-vector $\breve{u}_a$ represented in the non-fluactuating coordinates (e^a_μ, u_μ) such that

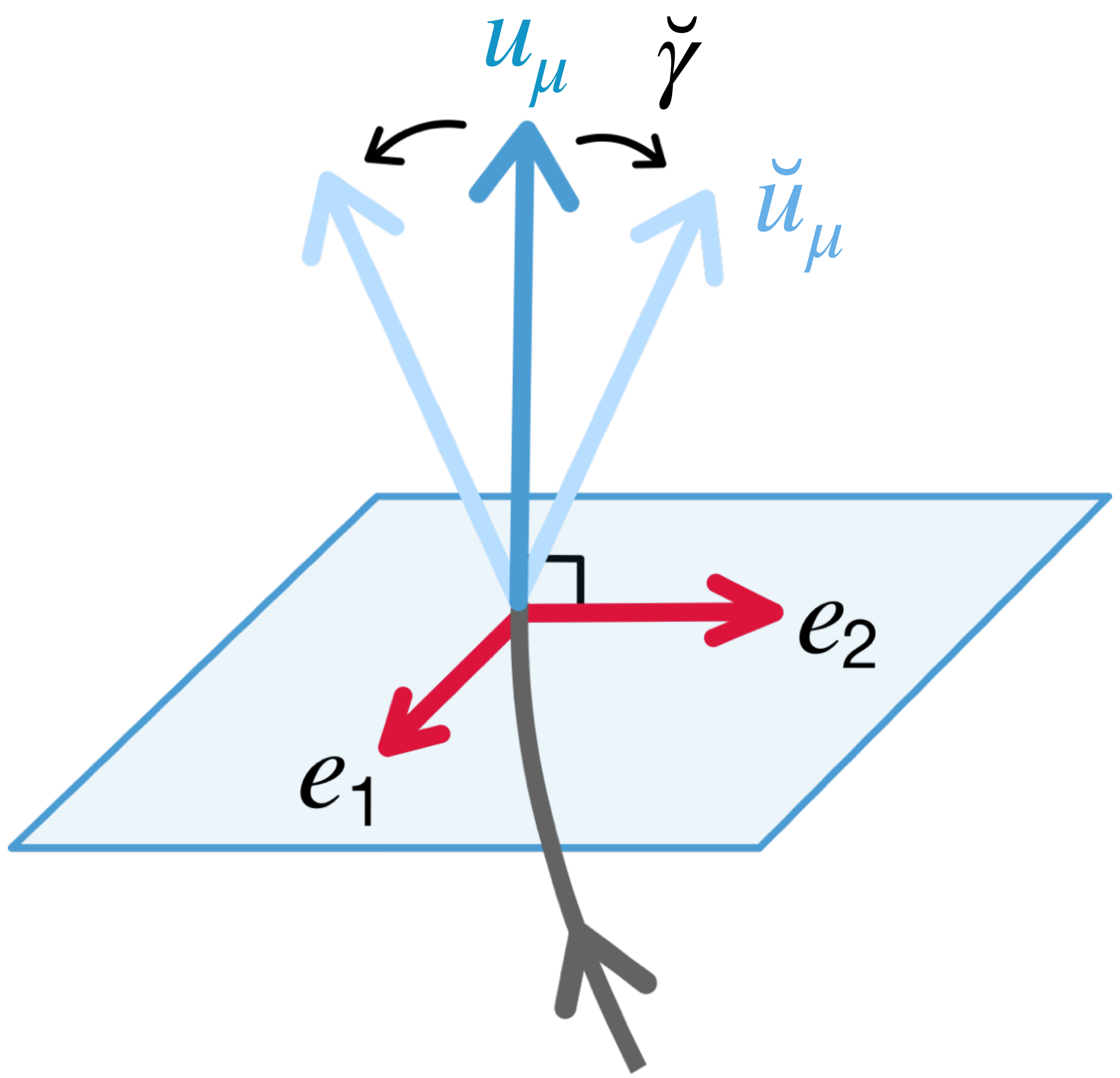
$$\langle \breve{u}_a \rangle = u_a \,.$$

$$\breve{u}_\mu = \breve{u}_a \, e^a_\mu + \breve{\gamma} \, u_\mu$$

velocity fluctuation $\breve{u}_\mu$ is measured in terms of its independent spatial components $\breve{u}_a$ in the LRF of u_μ, which is a comoving "LF" of $\breve{u}_\mu$, instead of an arbitrary *fixed* LF.

$$\breve{\gamma} \equiv \gamma(\breve{u}_a) = (1 + \breve{u}_a^2)^{1/2}$$

non-negligible non-linearity for non-Gaussian dynamics.



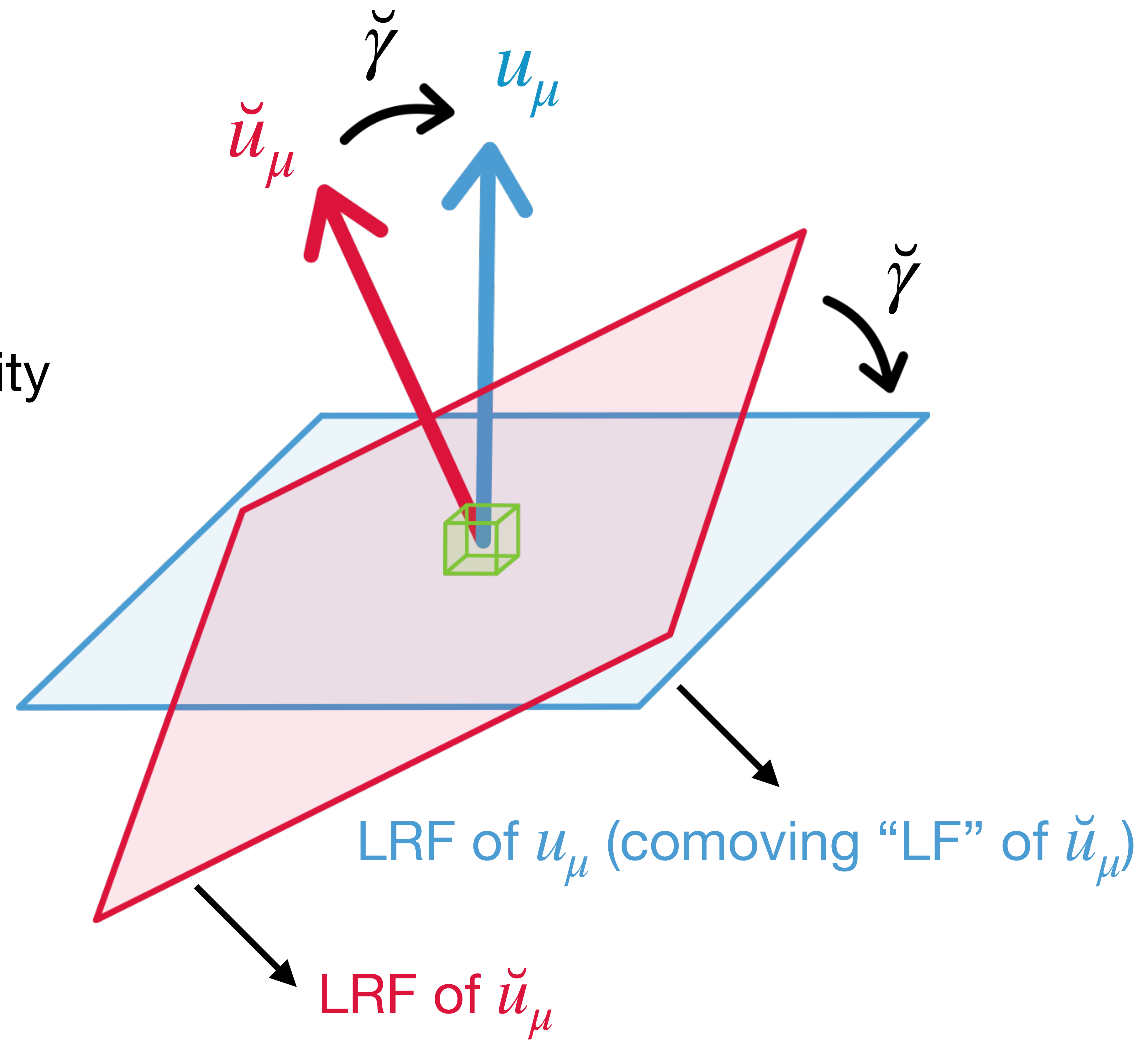
Entropy measured in the comoving "lab frame"

- Entropy is measured in the non-fluctuating LRF of u_μ in terms of fluctuating variables measured in the LRF of $\breve{u}_\mu$ (related by boost $\breve{\gamma}$).

$$S(\breve{\epsilon}, \breve{n}, \breve{u}_a) = \int_x \breve{\gamma}\breve{s} + \alpha\breve{\gamma}\breve{n} - \beta(\breve{\gamma}^2\breve{w} - \breve{p})$$

ϵ: energy density; n: charge density; u_a: three-velocity

where $\breve{\gamma} = (1 + \breve{u}_a^2)^{1/2}$, $\breve{w} = \breve{\epsilon} + \breve{p}$, $\breve{p} = p(\breve{\epsilon}, \breve{n})$,
α and β are Lagrange multipliers controlling the fluctuations of charge and energy respectively, in the LRF of u_μ.



Confluent fluctuation evolution equations

- Fluctuation evolution equations in the *impressionistic* form:

$$\mathscr{L}W_n = iqW_n - \gamma q^2(W_n - \ldots) - \partial\psi W_n + \ldots \quad \text{where} \quad \mathscr{L} = u\cdot\bar{\nabla}_x + f\cdot\nabla_q$$

sound/advection dissipation background

of which the solutions match results determined from entropy $S(\breve{m},\breve{p},\breve{u}_a)$.

m: entropy per baryon; p: pressure; u_a: three-velocity

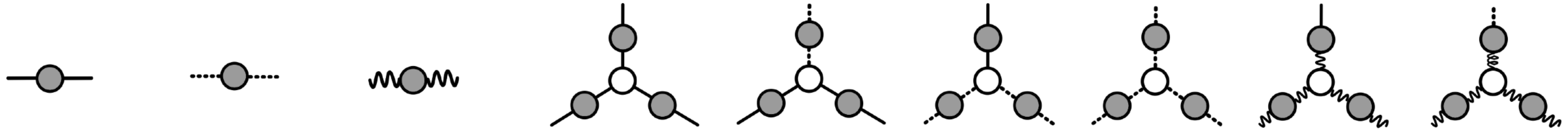
$$\mathscr{L}W_{ab}(\mathbf{q}_1,\mathbf{q}_2) = -\gamma_\eta(\mathbf{q}_1^2+\mathbf{q}_2^2)(W_{ab}-W_{ab}^{\text{eq}}) + \ldots;$$

$$\mathscr{L}W_{abc}(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3) = -\gamma_\eta(\mathbf{q}_1^2+\mathbf{q}_2^2+\mathbf{q}_3^2)W_{abc} + \ldots; \quad \ldots$$

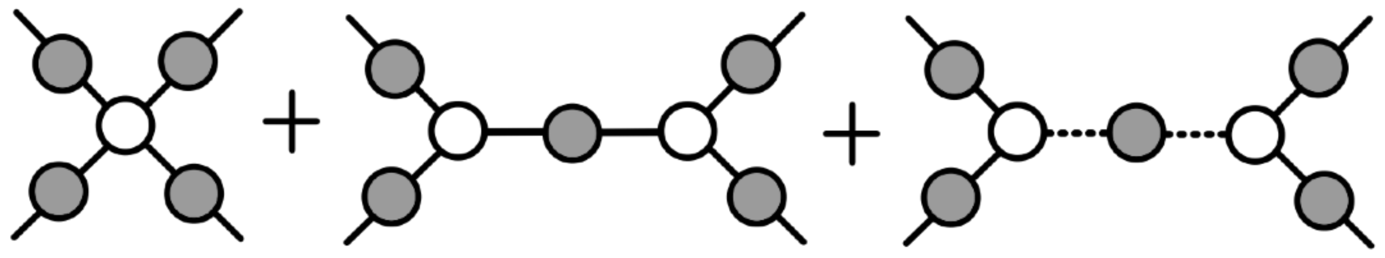
$$W_{ab}^{\text{eq}} = -(\beta w)^{-1}\delta_{ab}$$

$$W_{abc}^{\text{eq}} = 0$$

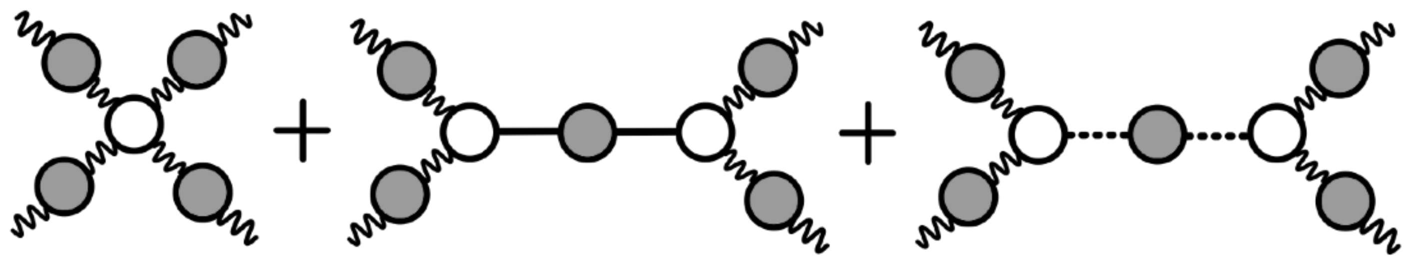
$$W_{abcd}^{\text{eq}} \sim -3(\beta w)^{-3}\delta_{ab}\delta_{cd}$$



W^{eq}_{mm} W^{eq}_{pp} W^{eq}_{ab} W^{eq}_{mmm} W^{eq}_{mmp} W^{eq}_{mpp} W^{eq}_{ppp} W^{eq}_{mab} W^{eq}_{pab}



W^{eq}_{mmmm}



W^{eq}_{abcd}

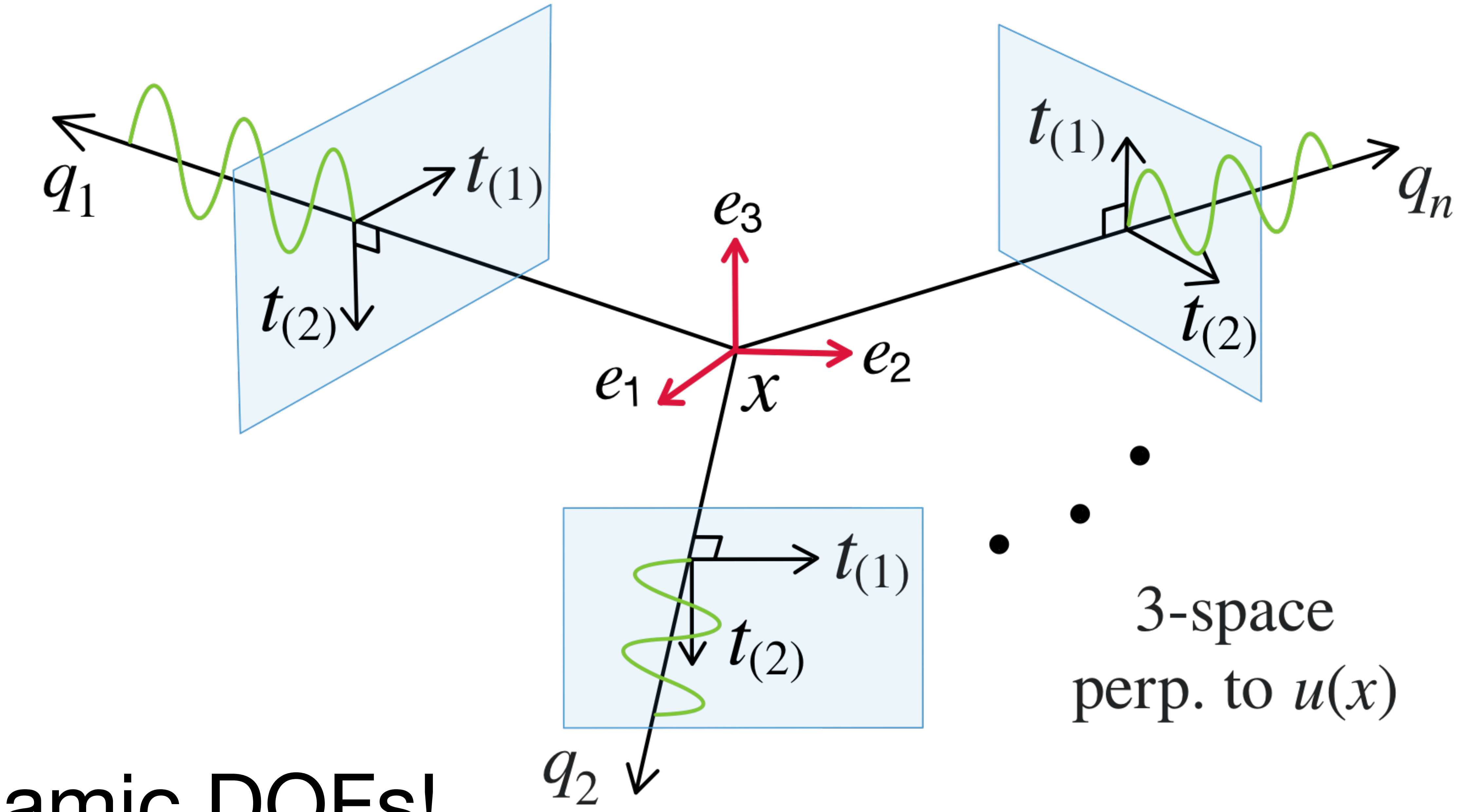
Rotating wave approximation

- We further introduce a local *spatial dyad* perpendicular to each $\mathbf{q}$, such that longitudinal velocity fluctuations decouple from their transverse partners.

$$\phi = \begin{pmatrix} \phi_m \\ \phi_p \\ \phi_a \end{pmatrix} = \begin{pmatrix} \delta m \\ \delta p \\ \delta u_a \end{pmatrix} \longrightarrow \Phi = \begin{pmatrix} \Phi_m \\ \Phi_\pm \\ \Phi_{(i)} \end{pmatrix} = \begin{pmatrix} \delta m \\ \delta p \pm c_s w \hat{\mathbf{q}}^a \delta u_a \\ t^a_{(i)} \delta u_a \end{pmatrix} \qquad (i) = 1, 2$$

the "sound-front" basis with 5 eigenvalues
$\lambda_\pm(\mathbf{q}) = \pm c_s |\mathbf{q}|, \; \lambda_m(\mathbf{q}) = \lambda_{(i)}(\mathbf{q}) = 0$

$$\mathscr{L} W_{\Phi_1 \ldots \Phi_n} = \left(\sum_{i=1}^{n} \lambda_{\Phi_i}(\mathbf{q}_i) \right) W_{\Phi_1 \ldots \Phi_n} + \ldots$$



- In the "sound-front" basis RWA says

$$\text{if } \sum_{i=1}^{n} \lambda_{\Phi_i}(\mathbf{q}_i) \begin{cases} = 0 & \longrightarrow \quad \text{slow mode (kept)} \\ \neq 0 & \longrightarrow \quad \text{fast mode (averaged)} \end{cases}$$

a significant reduction of *independent* dynamic DOFs!

RWA equations

- We end up with 1 equation for W_{+-}, and other 12 (diagrammatically different) equations with $A = (m, (i))$ up to 4-pt: in progress

$$\mathscr{L}[W_{A_1A_2}(\mathbf{q}_1, \mathbf{q}_2)] = 2\left[L_{A_1,B_1}(\mathbf{q}_1,)W_{B_1A_2}(,\mathbf{q}_2) + Q_{A_1A_2}(\mathbf{q}_1, \mathbf{q}_2)\right]_{\overline{12}},$$

$$\mathscr{L}[W_{A_1A_2A_3}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)]$$
$$= 3\left[L_{A_1,B_1}(\mathbf{q}_1,)W_{B_1A_2A_3}(,\mathbf{q}_2, \mathbf{q}_3) + L_{A_1,B_1B_2}(\mathbf{q}_1,,)W_{B_1A_2}(,\mathbf{q}_2)W_{B_2A_3}(,\mathbf{q}_3) + 2Q_{A_1A_2,B_1}(\mathbf{q}_1, \mathbf{q}_2,)W_{B_1A_3}(,\mathbf{q}_3)\right]_{\overline{123}},$$

$$\mathscr{L}[W_{A_1A_2A_3A_4}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4)]$$
$$= 4\left[L_{A_1,B_1}(\mathbf{q}_1,)W_{B_1A_2A_3A_4}(,\mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) + 3L_{A_1,B_1B_2}(\mathbf{q}_1,,)W_{B_1A_2}(,\mathbf{q}_2)W_{B_2A_3A_4}(,\mathbf{q}_3, \mathbf{q}_4)\right.$$
$$+L_{A_1,B_1B_2B_3}(\mathbf{q}_1,,,)W_{B_1A_2}(,\mathbf{q}_2)W_{B_2A_3}(,\mathbf{q}_3)W_{B_3A_4}(,\mathbf{q}_4)$$
$$\left.+3Q_{A_1A_2,B_1}(\mathbf{q}_1, \mathbf{q}_2,)W_{B_1A_3A_4}(,\mathbf{q}_3, \mathbf{q}_4) + 3Q_{A_1A_2,B_1B_2}(\mathbf{q}_1, \mathbf{q}_2,,)W_{B_1A_3}(,\mathbf{q}_3)W_{B_2A_4}(,\mathbf{q}_4)\right]_{\overline{1234}},$$

numerical implementation welcome!

Recap

- Velocity fluctuations matter in HIC (collectivity and criticality).
- The incorporation of velocity fluctuations into the general deterministic formalism for non-Gaussian fluctuations is challenging, but can be done systematically.

Outlook

- Establish quantitative connection between parametrized EOS and experiment, taking into account the non-equilibrium evolution (including flow and its fluctuations).
- Use BES-II data to constrain the EOS and transport coefficients.

Thank You!