Lee-Yang singularities, series expansions and the critical point

Gökçe Başar

University of North Carolina, Chapel Hill

CPOD 24, Lawrence Berkeley Lab, May 20-24, 2024

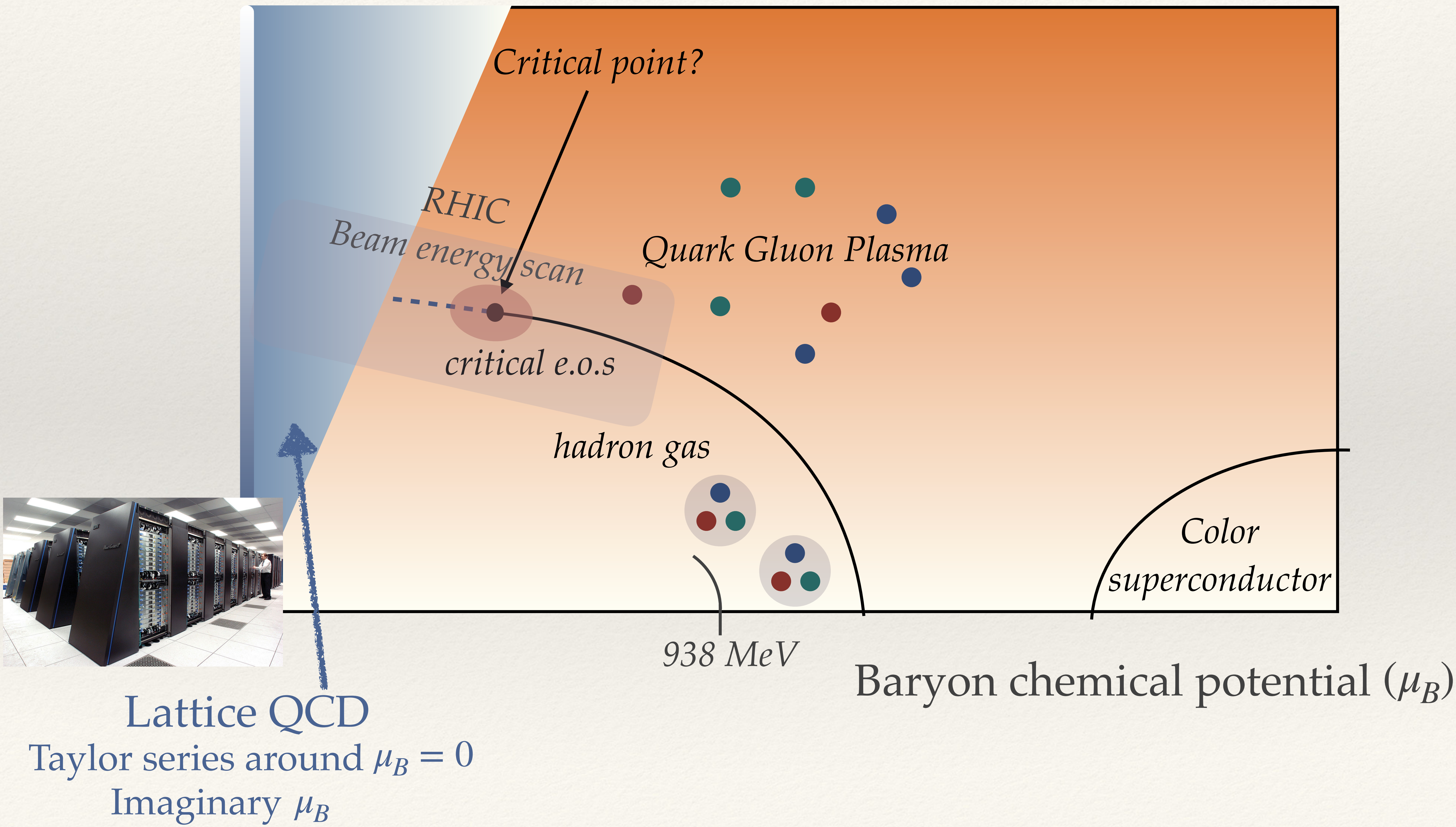
Based on:

GB, 2312.06952

GB, *PRL* 127 (2021) 17, 171603

GB, G. Dunne, Z. Yin *PRD* 105 (2022) 10, 105002

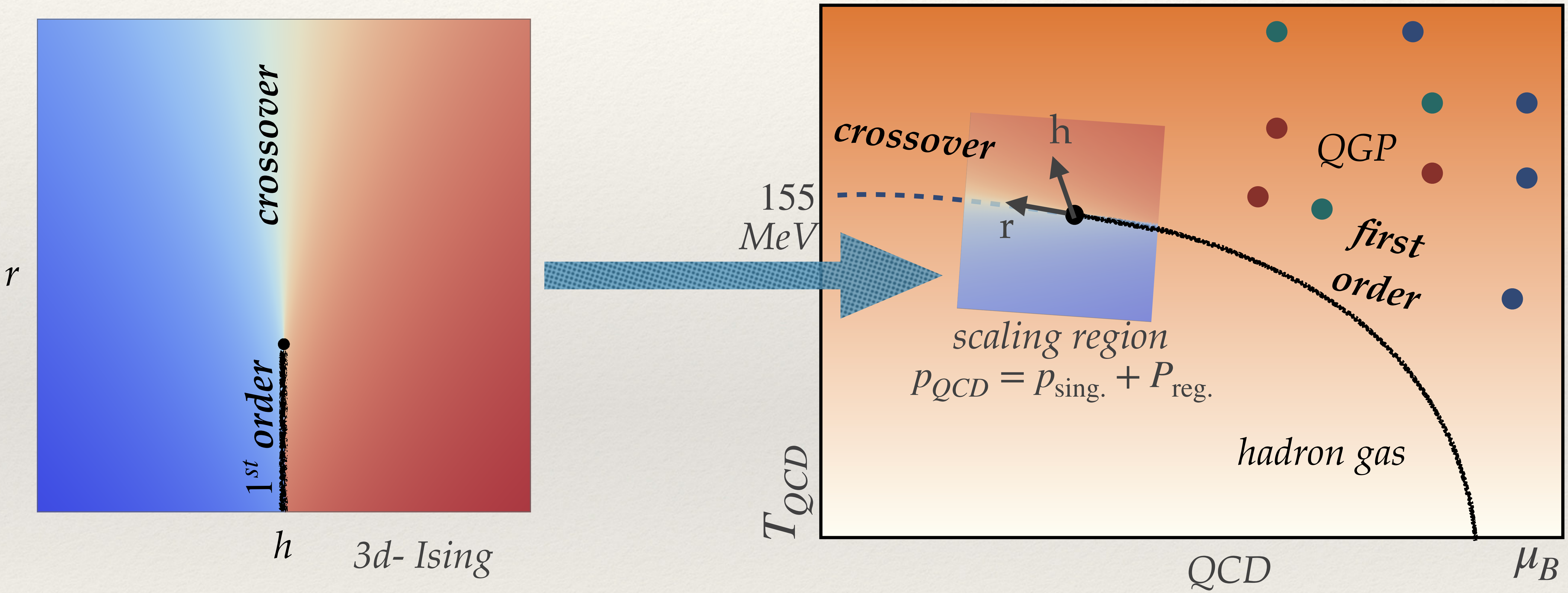
Motivations



Motivations

$$\begin{pmatrix} r \\ h \end{pmatrix} = \begin{pmatrix} r_T & r_\mu \\ h_T & h_\mu \end{pmatrix} \begin{pmatrix} T - T_c \\ \mu - \mu_C \end{pmatrix} := M \begin{pmatrix} T - T_c \\ \mu - \mu_C \end{pmatrix}$$

[see talk by Kahangirwe]

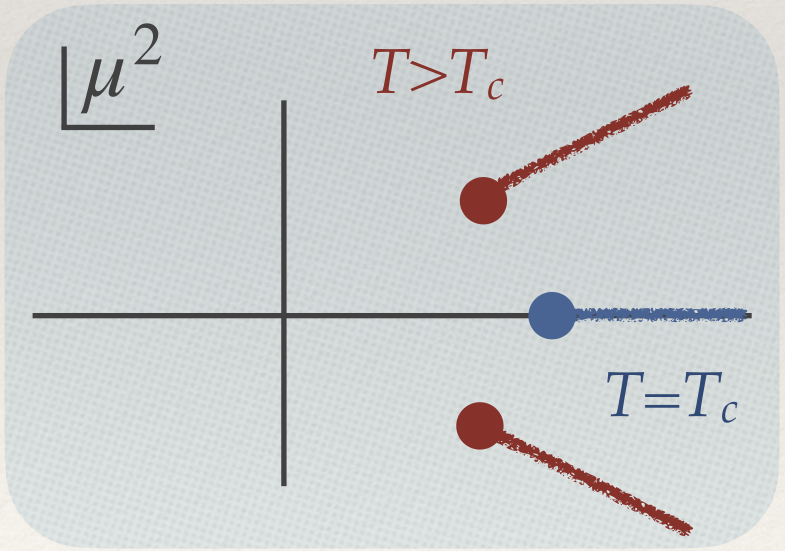


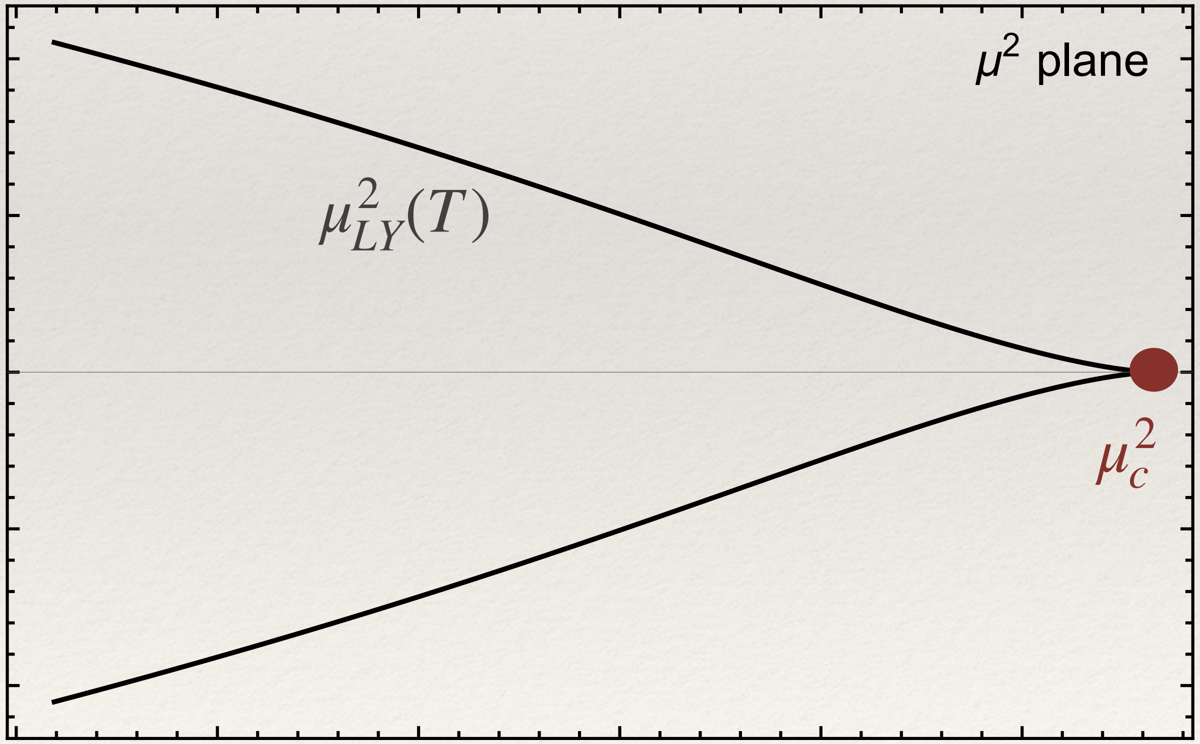
Given the e.o.s. as truncated Taylor series around μ=0, what can we say about *the critical e.o.s* ?

Lee-Yang edge singularities

- The equation of state has complex singularities [Lee-Yang, 52′]
- Zeroes of partition function $\mathcal{Z}(\zeta)$ ($\zeta = e^{\mu/T}$: fugacity)
- Coalesce into branch cuts in thermodynamic limit
- Pinch the real axis at a second order transition
- Closest singularity to origin (“extended analyticity conjecture”)

[Fonseca, Zamolodchikov ’02, An, Mesterházy, Stephanov ’17]





[Stephanov, 0603014]

Lee Yang edge singularity

- The scaling e.o.s, $f_s(w)$, has singularities at $w = \pm iw_{LY}$ $(w := hr^{-\beta\delta})$

$$\mu_{LY}(T) \approx \mu_c - \frac{h_T}{h_\mu}(T - T_c) \pm iw_{LY}\frac{\left(\det \mathbb{M}\right)^{\beta\delta}}{h_\mu^{\beta\delta+1}}(T - T_c)^{\beta\delta}$$

[Stephanov '06]

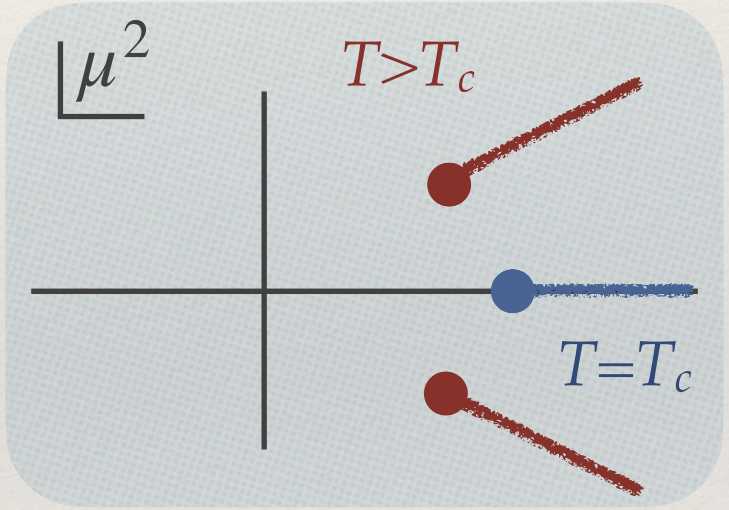
$(\tan\alpha_1)^{-1}$

slope of the crossover line

$\det \mathbb{M} \propto (\tan\alpha_2 - \tan\alpha_1)$

relative angle between r, h axes

see [Pradeep, Stephanov '19]



- The e.o.s. near the LY singularity: $M(w) \sim (w \pm iw_{LY})^{\sigma_{LY}}$, $(M$: magnetization)

$\sigma_{LY,d=3} \approx 0.1$, $\sigma_{LY,d=6} = 1/2$ (mean field)

[Fisher, '74; An, Stephanov, Mesterházy '16; Connelly, Johnson, Mukherjee, Skokov '20]

When life gives you Taylor series...

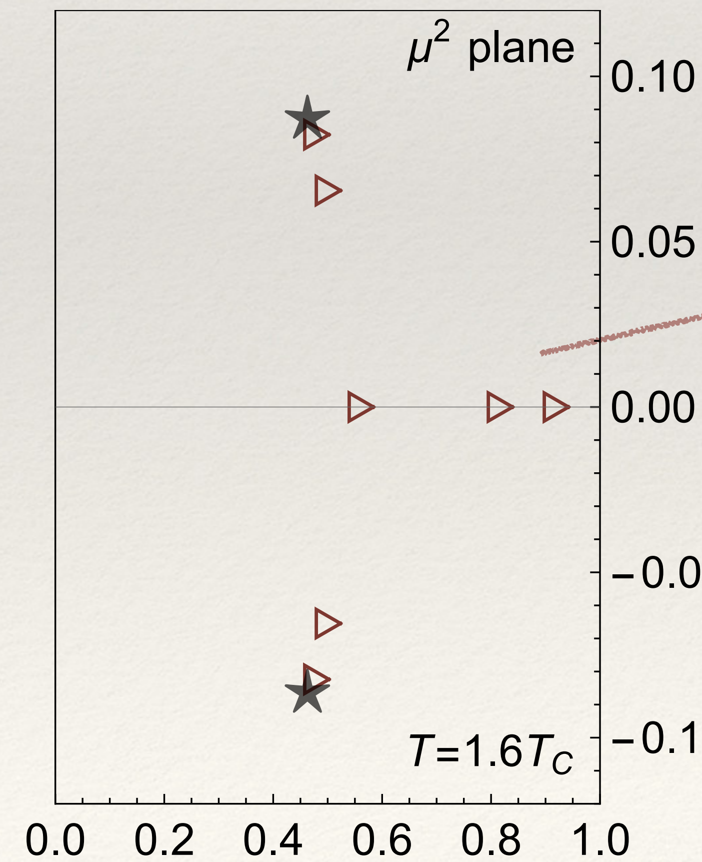
Taylor series: $\chi(\mu^2) = \sum_{n=0}^{N} c_{2n}\mu^{2n}$

Padé approximant (diagonal) $\mathrm{P}_{[N/2,N/2]}f(\mu^2) = \dfrac{P_{N/2}(\mu^2)}{Q_{N/2}(\mu^2)}$

Singularity of the function ⟷ *poles/zeroes of Padé*

e.g. GN model

Problem: Padé is fairly good away from the singularity but fails badly near a singularity / branch cut 🤷‍♂️

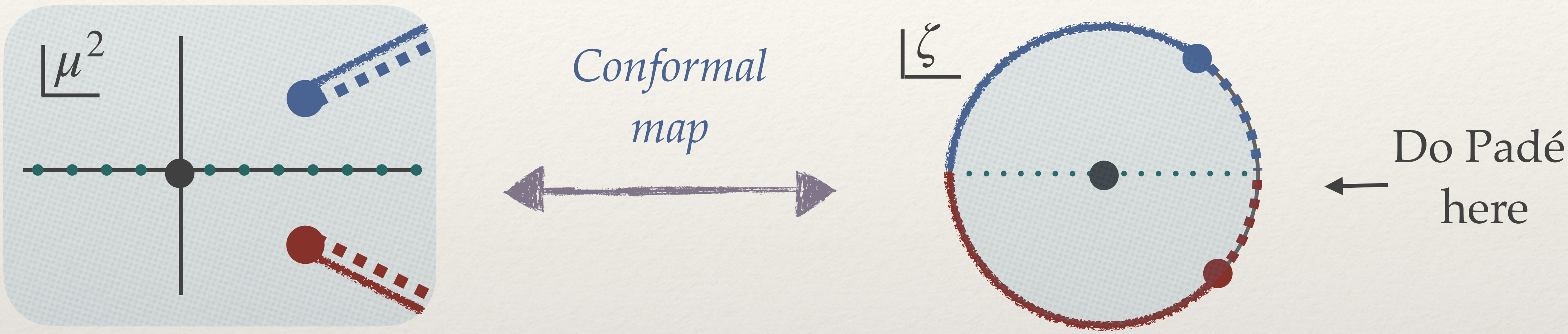


Unphysical poles

[Stahl' 97, Costin Dunne '20]

Conformal Maps

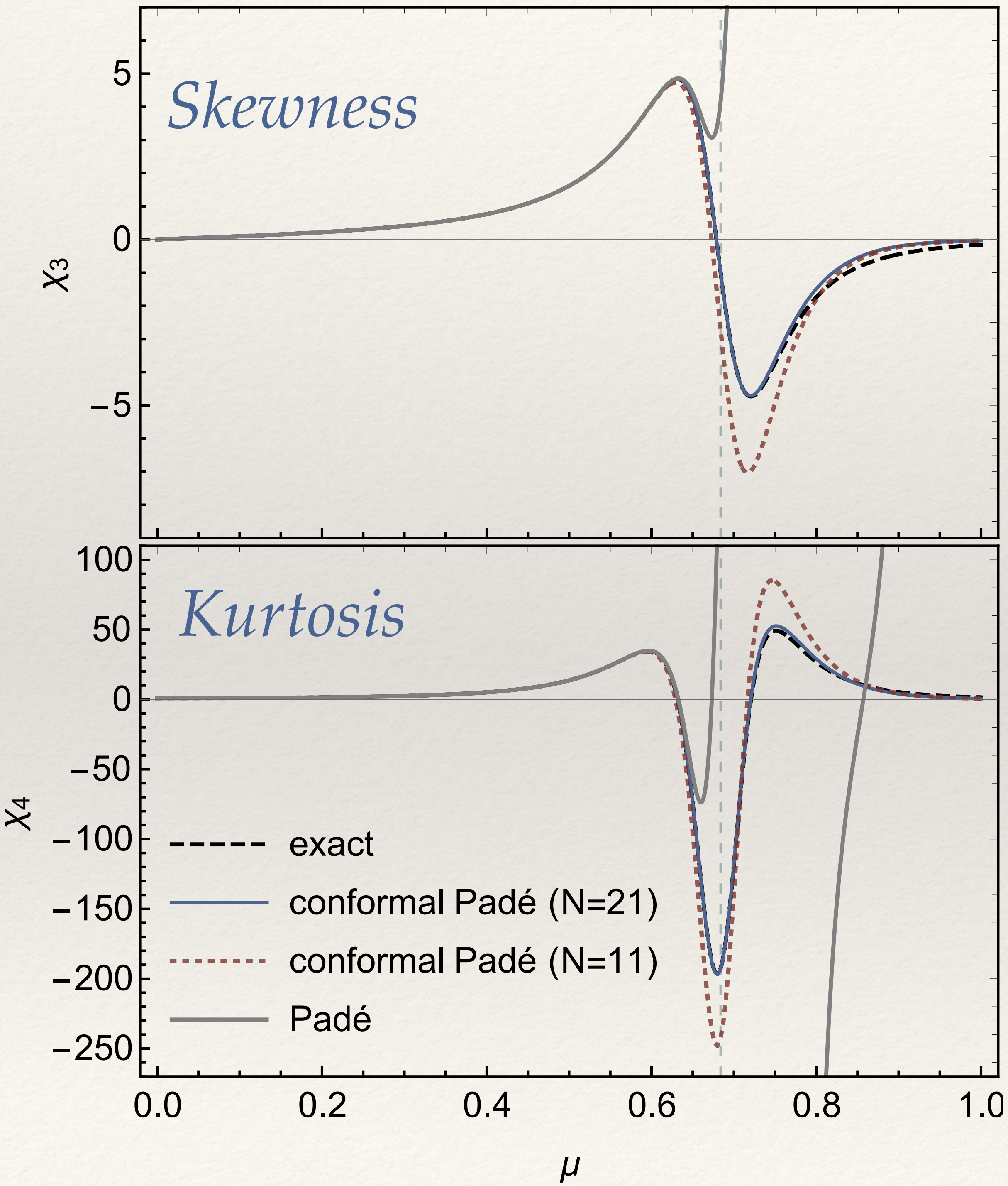
Solution: Do Padé after a conformal map



"conformal Padé" $$P\chi(T,\phi(\zeta)) = \left.\frac{\tilde{p}_0(T) + \tilde{p}_1(T)\zeta + \ldots + \tilde{p}_{N/2}(T)\zeta^N}{\tilde{q}_0(T) + \tilde{q}_1(T)\zeta + \ldots + \tilde{q}_{N/2}(T)\zeta^N}\right|_{\zeta=\phi^{-1}(\mu^2)}$$

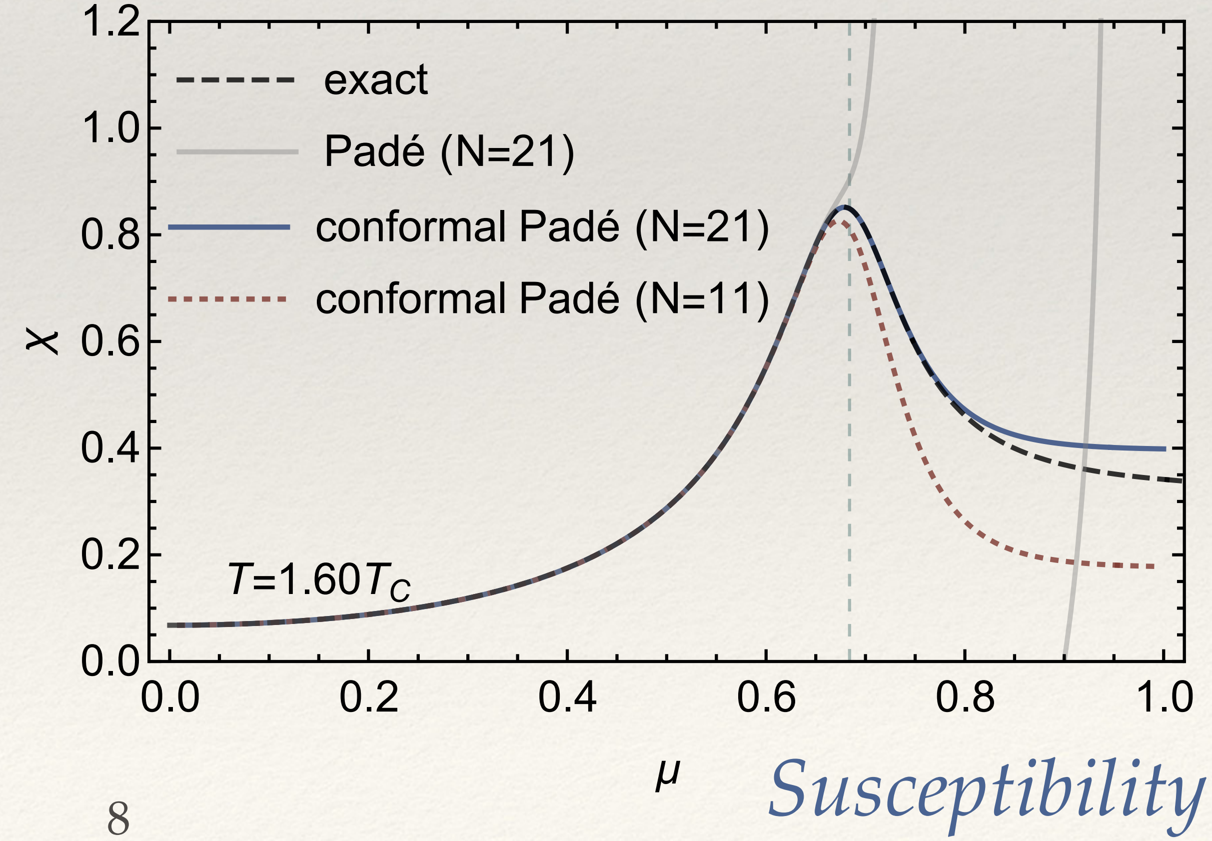
- Captures the singular behavior, no unphysical poles along real axis
- Significantly better approximation than Padé
- Can go beyond the radius of convergence, even to different Riemann sheets!

Conformal Maps



Test case: GN model

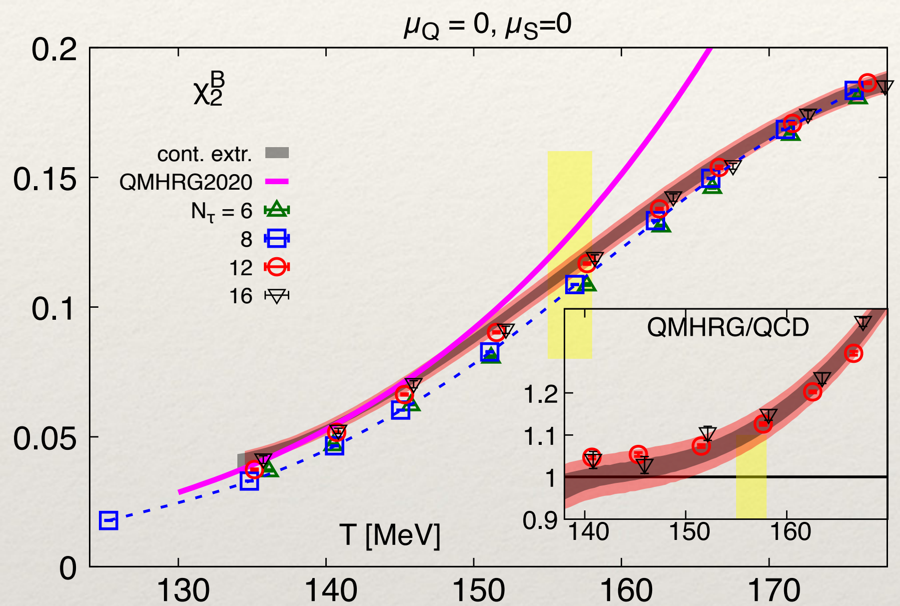
- conformal Padé does not introduce unphysical poles on the real axis!
- captures the e.o.s. *beyond the radius of convergence*

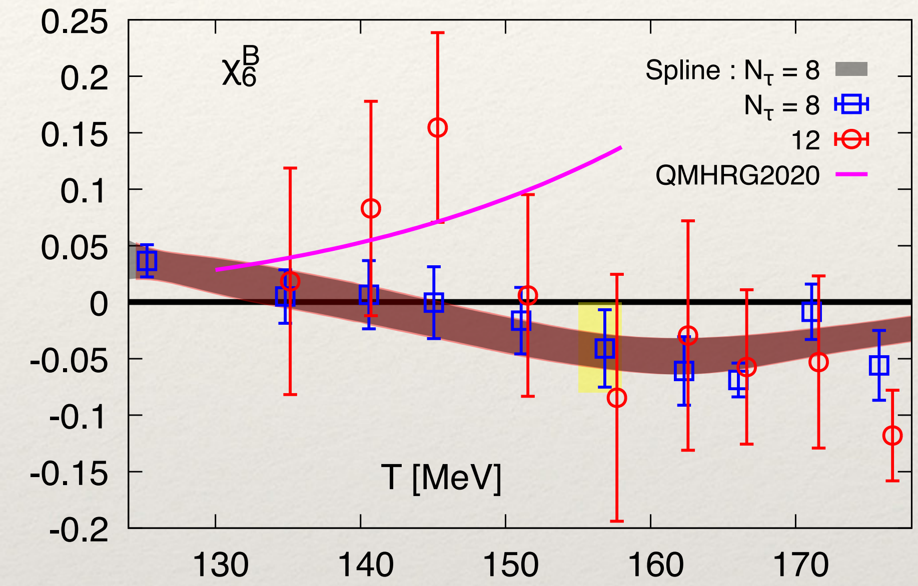


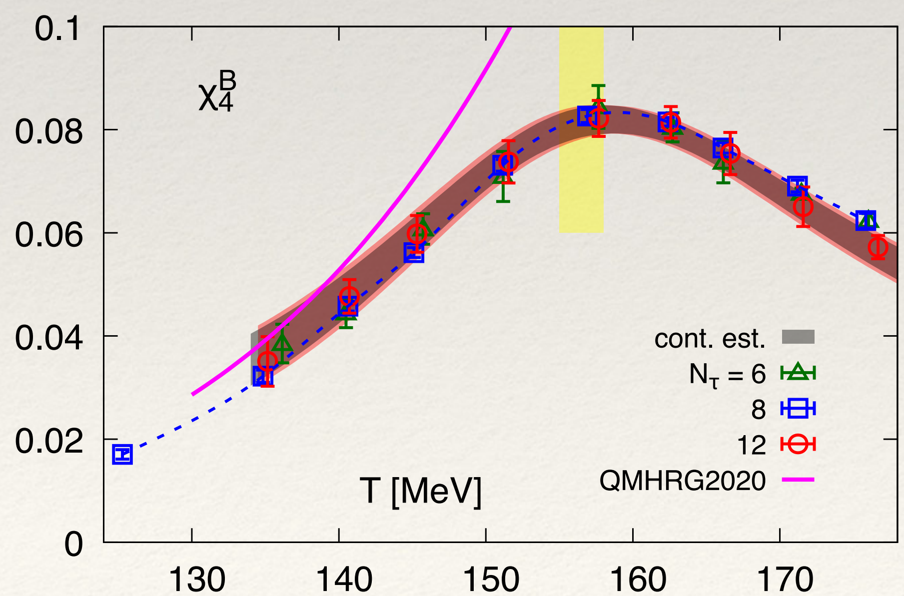
Taylor coefficients for QCD (HotQCD)

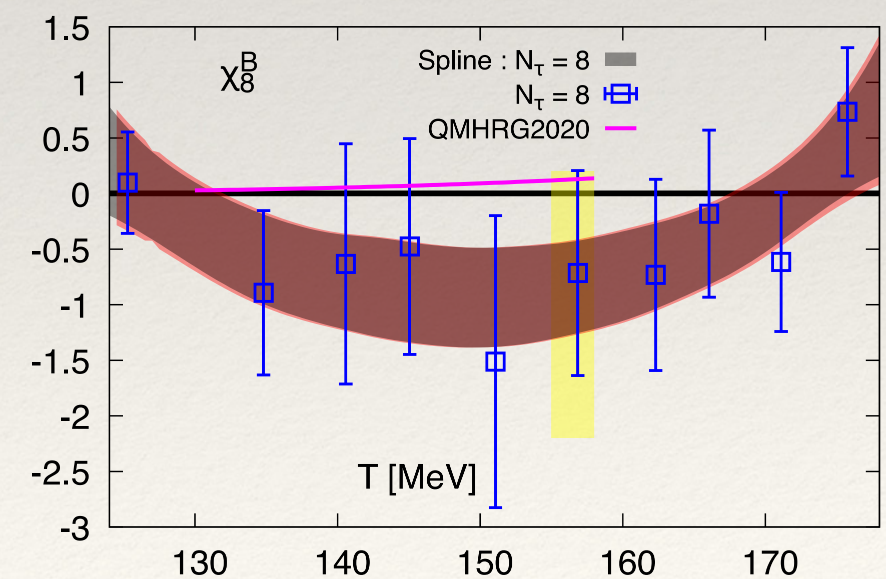
Taylor coefficients from Hot QCD collaboration up to μ_B^8

[Bollweg et al. PRD 105 (2022) 7, 074511]





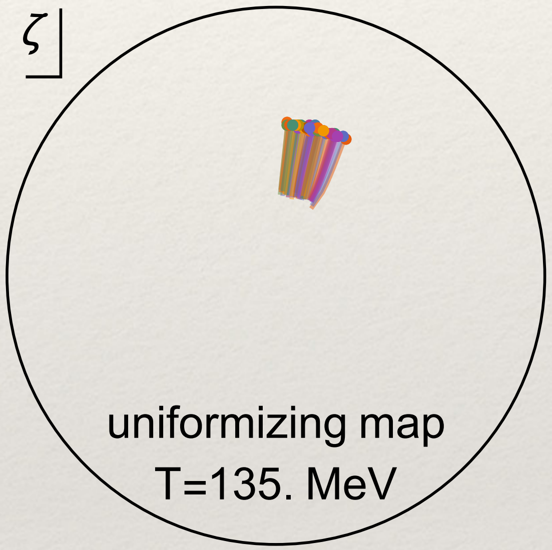


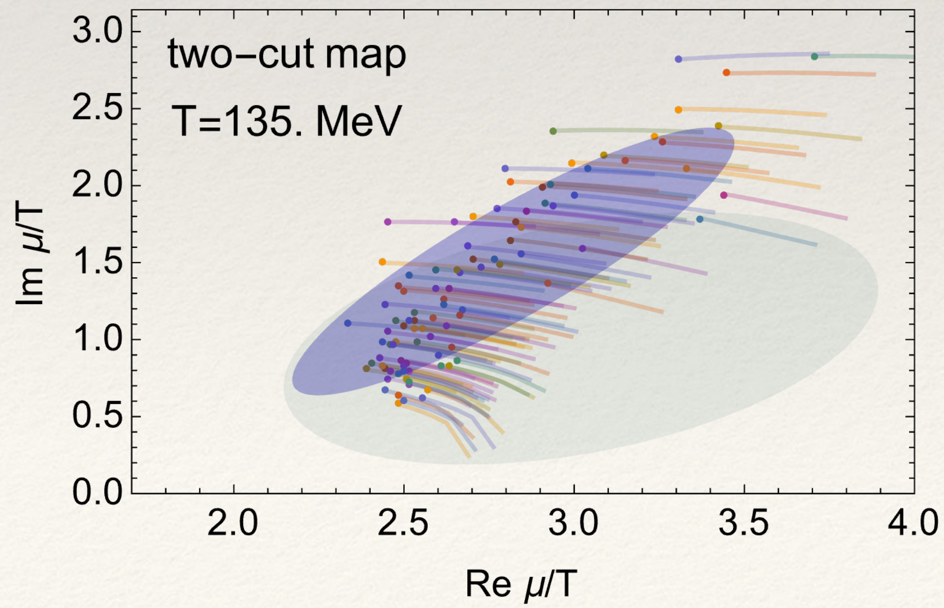


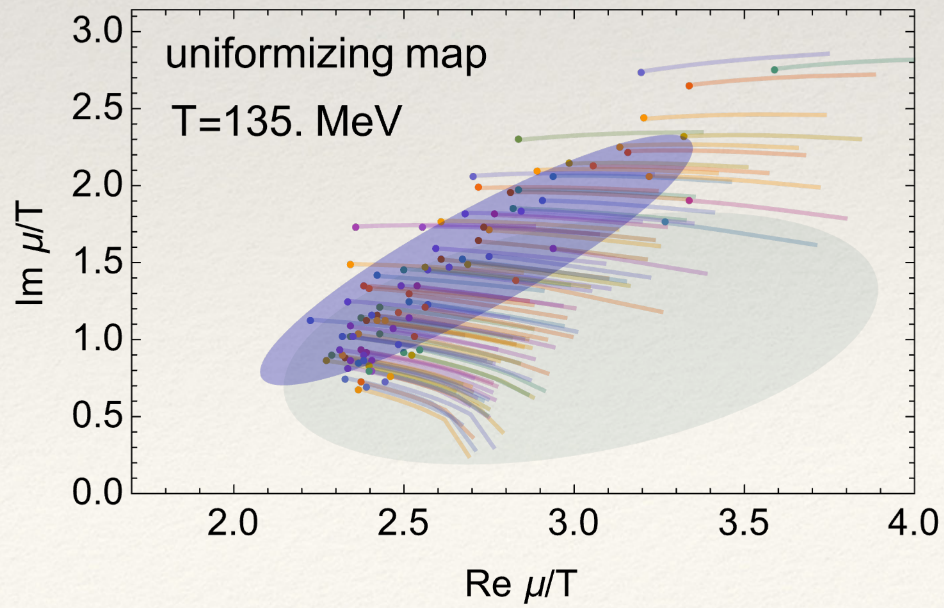
Conformal Pade Algorithm

- Sample Taylor coefficients from a Gaussian ensemble
- Estimate singularity from Pade as an input for conformal map
- Refine the estimate via conformal Pade
- Use the refined value in conformal map
- Repeat

Consistency check:
Estimates of the singularities approach the edge of the unit disk!

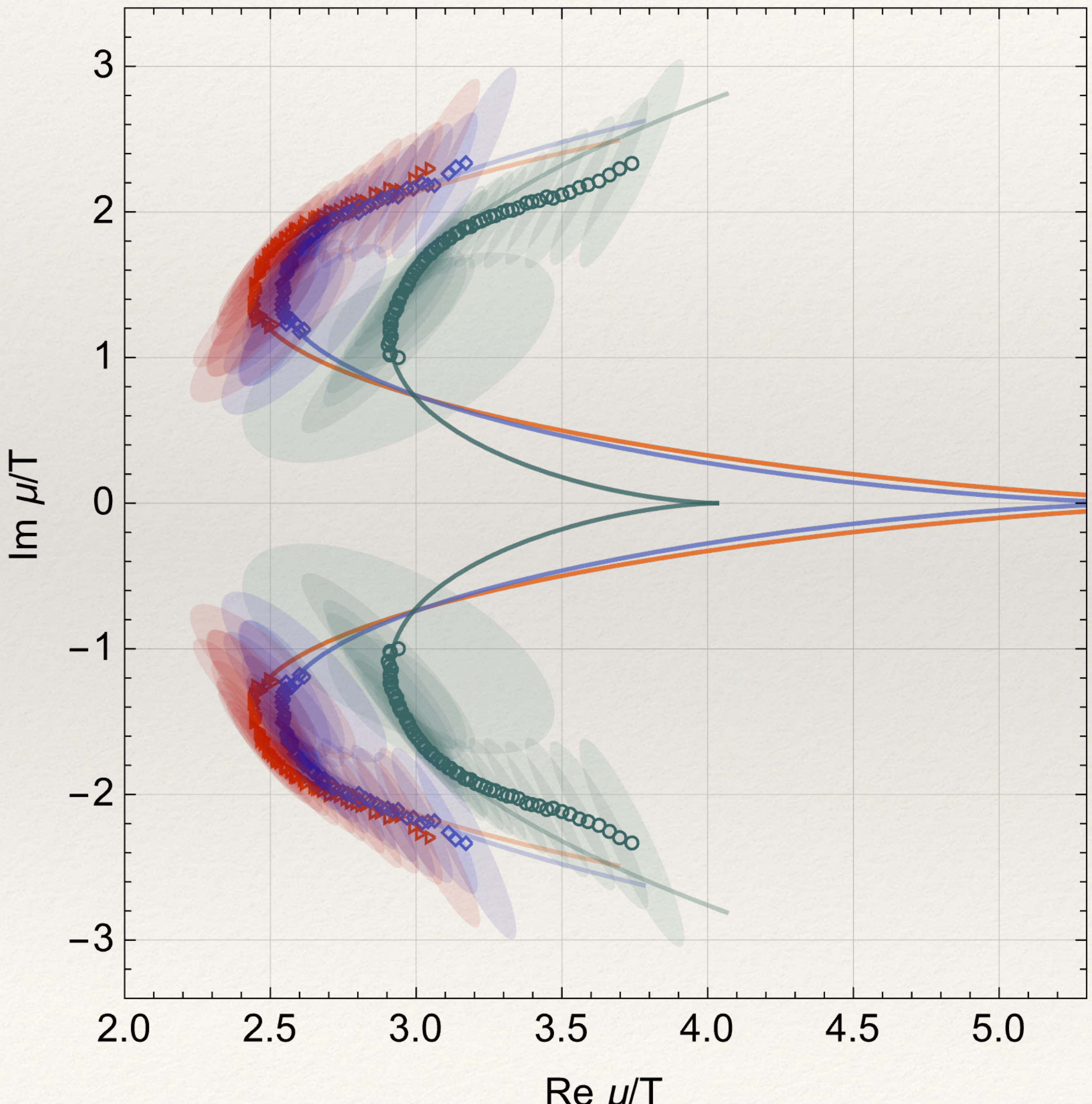






Lee Yang Trajectory

$$\mu_{LY}(T) \approx \mu_c - \frac{h_T}{h_\mu}(T - T_c) + i w_c \frac{r_\mu^{\beta\delta}}{h_\mu}\left(\frac{r_T}{r_\mu} - \frac{h_T}{h_\mu}\right)^{\beta\delta}(T - T_c)^{\beta\delta}$$



fits:

$$\mathrm{Re}\mu_{LY}(T) = a(T - T_C)^2 + b(T - T_C) + c$$

$$\mathrm{Im}\mu_{LY}(T) = c w_c (T - T_C)^{\beta\delta}$$

$\beta\delta \approx 1.5631$ (3d Ising)

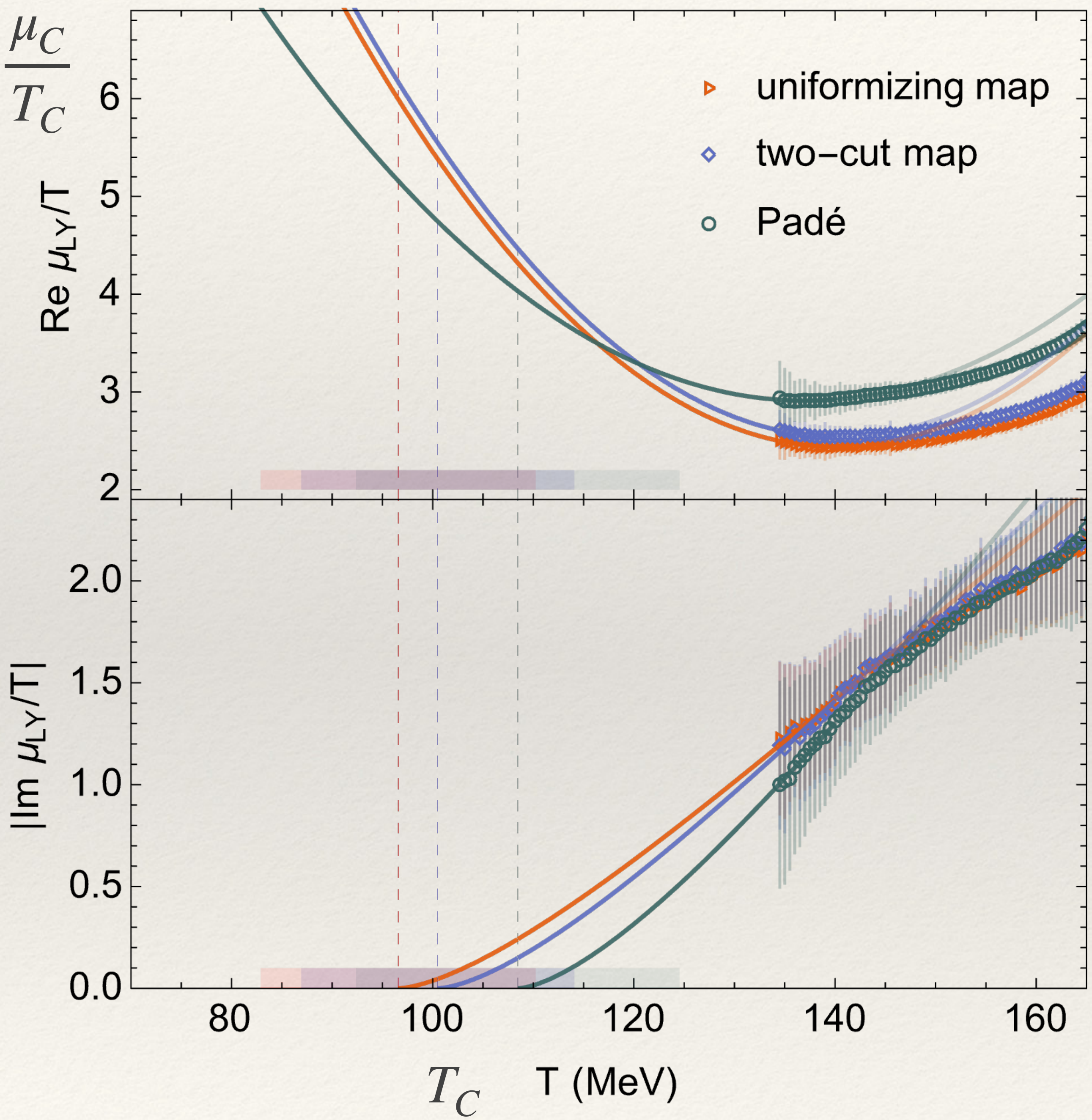
from conformal bootstrap
[Simmons-Duffin, 1502.02033]

$w_c = |z_c|^{-\beta\delta} \approx 0.246$

from functional RG
[Connelly et al, 2006.12541]

consistent with the HotQCD results
[Bollweg et al. 2202.09184]

Estimations of QCD critical point



unif. Padé $T_C = 97\,\text{MeV} \quad \mu_C = 579\,\text{MeV}$

$\alpha_1 = 9.40° \quad c = 2.22$

2-cut conf. Padé $T_C = 100\,\text{MeV} \quad \mu_C = 557\,\text{MeV}$

$\alpha_1 = 8.69° \quad c = 2.65$

Padé $T_C = 108\,\text{MeV} \quad \mu_C = 437\,\text{MeV}$

$\alpha_1 = 4.55° \quad c = 3.35$

1 sigma uncertainty:

$T_c :\sim \pm 20\text{MeV}, \quad \mu_c :\sim \pm 200\text{MeV}$

Bielefeld-Parma $T_C \sim 90\,\text{MeV} \quad \mu_C \sim 600\,\text{MeV}$

[Di Renzo, Clarke, Dimopoulos, Goswami, Schmidt '23 Lattice 23]

Functional RG $T_C \sim 107\,\text{MeV} \quad \mu_C \sim 635\,\text{MeV}$

[Fu, Pawlowski, Rennecke '20 PRD 101 054032]

Dyson-Schwinger: $T_C \sim 117\,\text{MeV} \quad \mu_C \sim 600\,\text{MeV}$

[Gunkel, Fischer 21. PRD 104 054022]

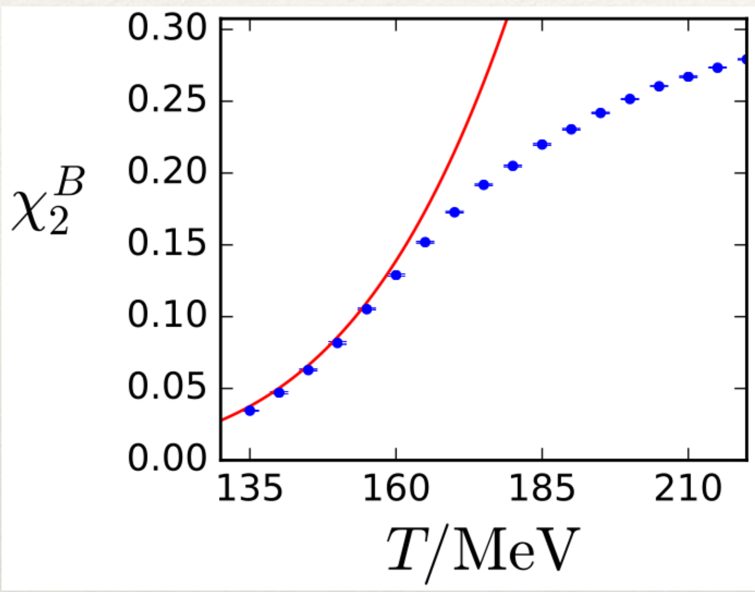
Holography: $T_C \sim 104\,\text{MeV} \quad \mu_C \sim 590\,\text{MeV}$

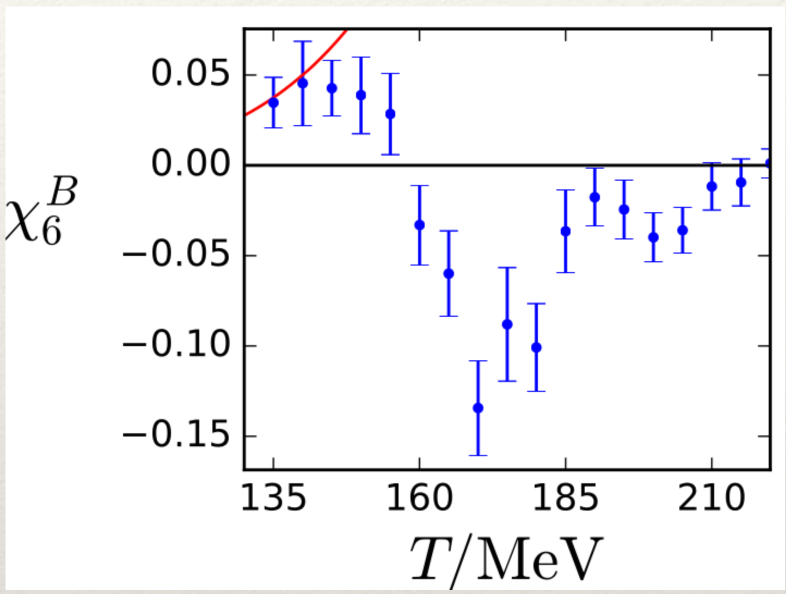
[Hippel et al 2309.00579]

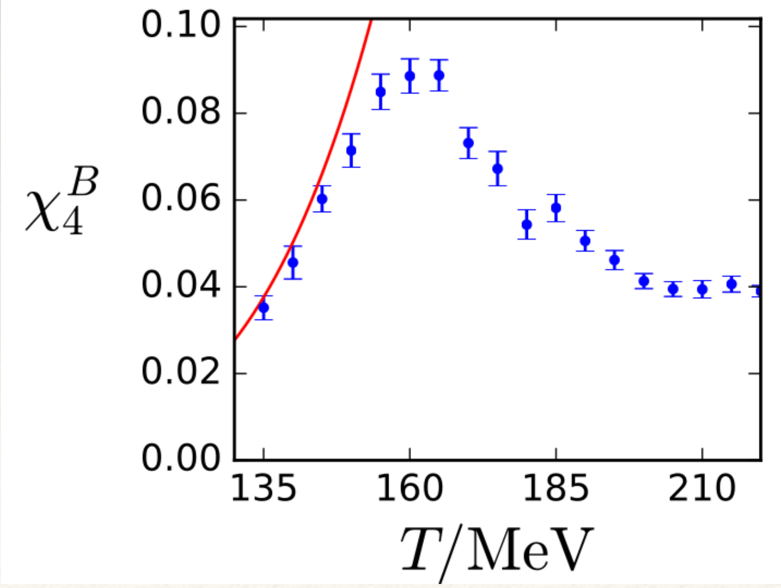
[see talks by Schmidt, Fisher, Noronha]

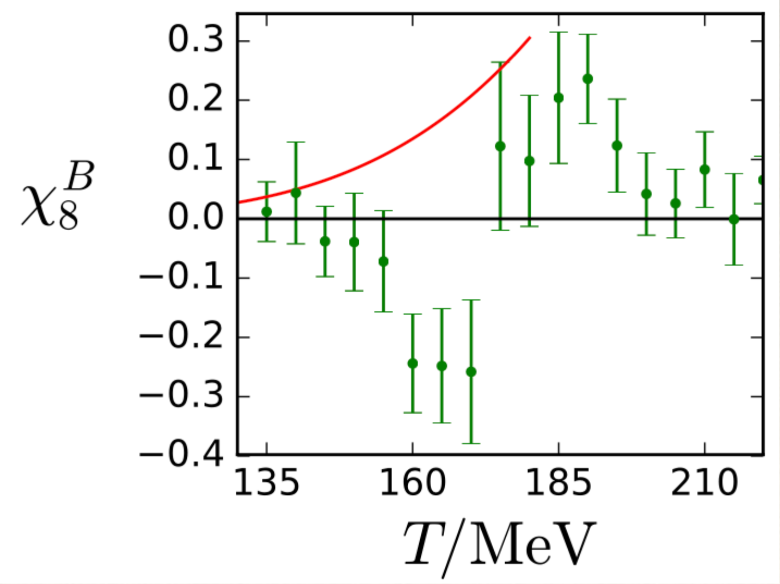
Taylor coefficients for QCD (Wuppertal–Budapest)

Taylor coefficients from Wuppertal-Budapest collaboration
[Borsanyi et al. JHEP 10(2018) 205]



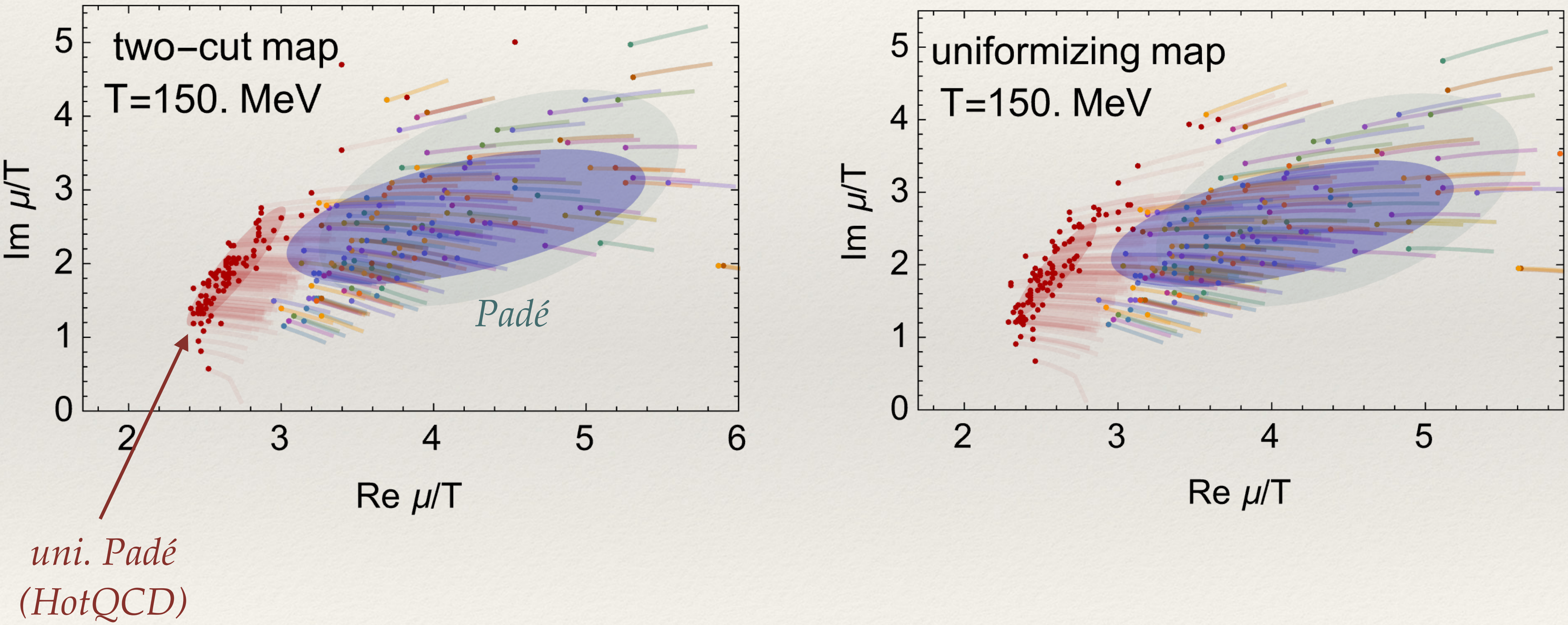




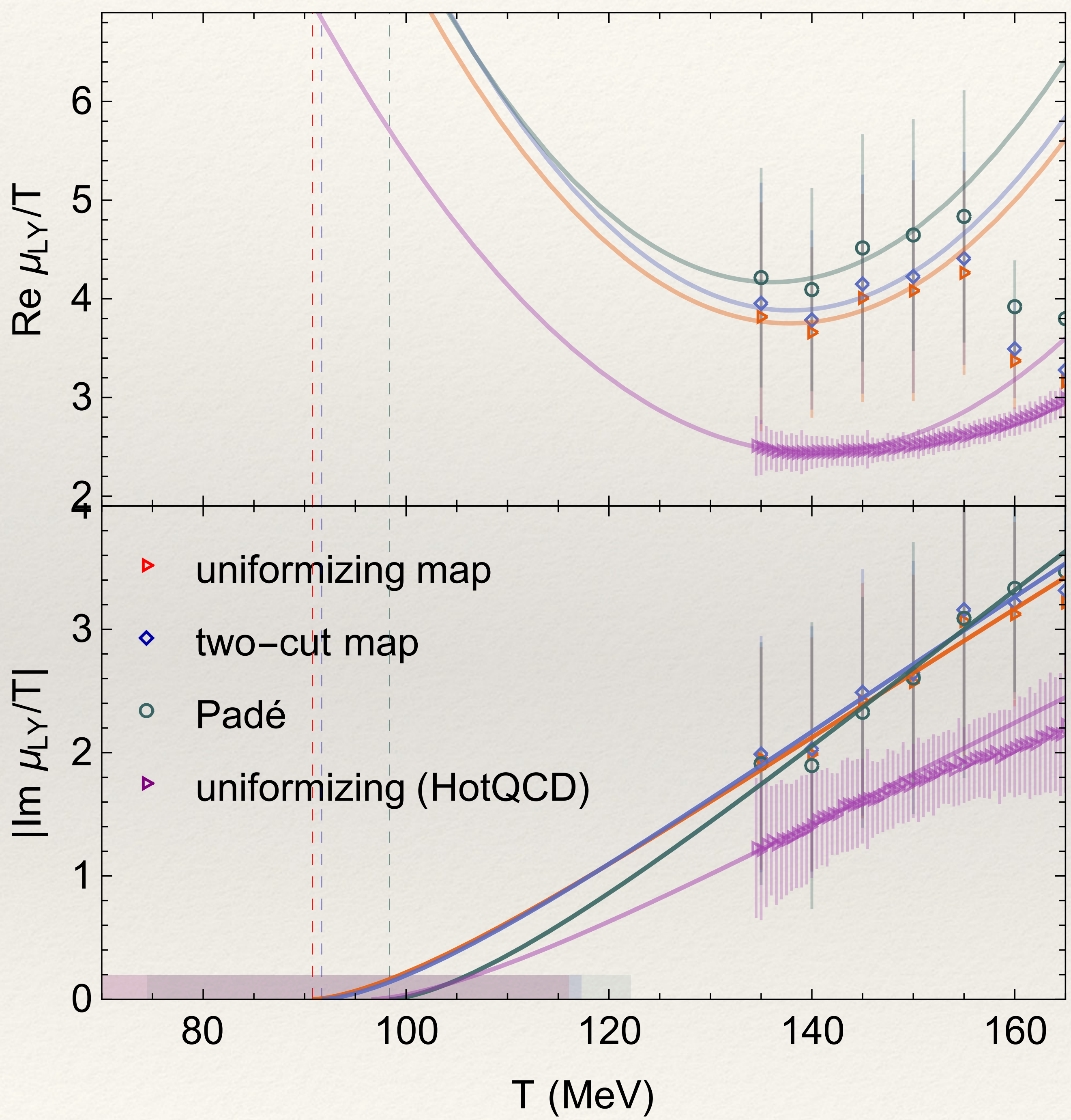


Estimations for QCD critical point from WP data

Iteration:



Estimations for QCD critical point from WP data



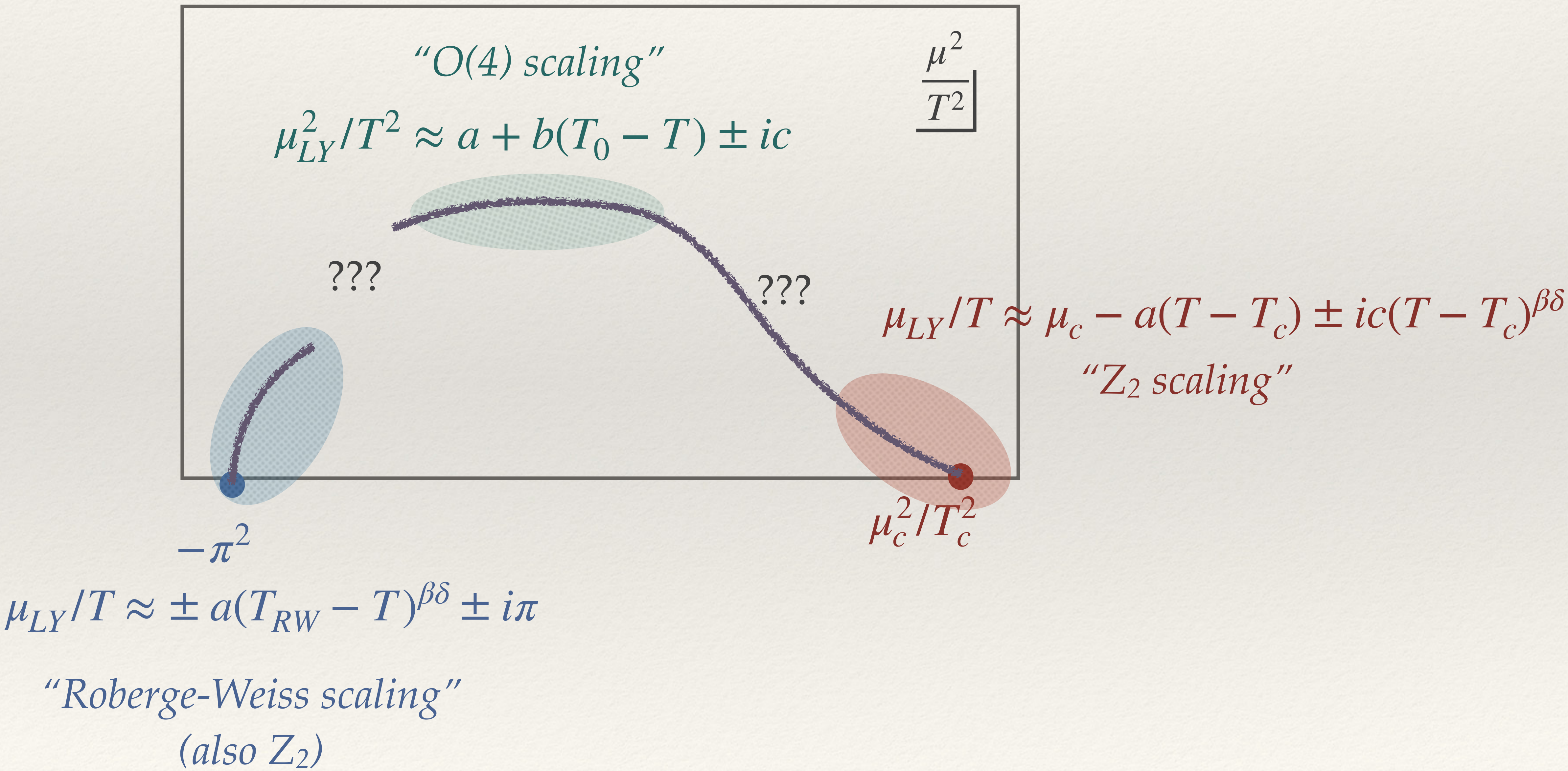
	unif. Padé	*2-cut Padé*	*Padé*
T_C (MeV) :	90	91	98
c :	*2.7*	*2.9*	*3.4*

1 sigma uncertainty T_C: ±25 MeV

statistical uncertainties are too large to estimate μ_C or the slope

A parade of singularities

A cartoon for Lee-Yang trajectory for QCD …



Conclusions and Outlook

- Combined with conformal maps, Padé approximants provide a powerful tool to estimate T_c , μ_c from a truncated Taylor series

- It is encouraging that these estimates agree with other methods

- The extrapolation of μ_c depends sensitively on the fit for $\mathrm{R}e\mu_{LY}$ and has a large uncertainty

- Lower T data would significantly improve the situation

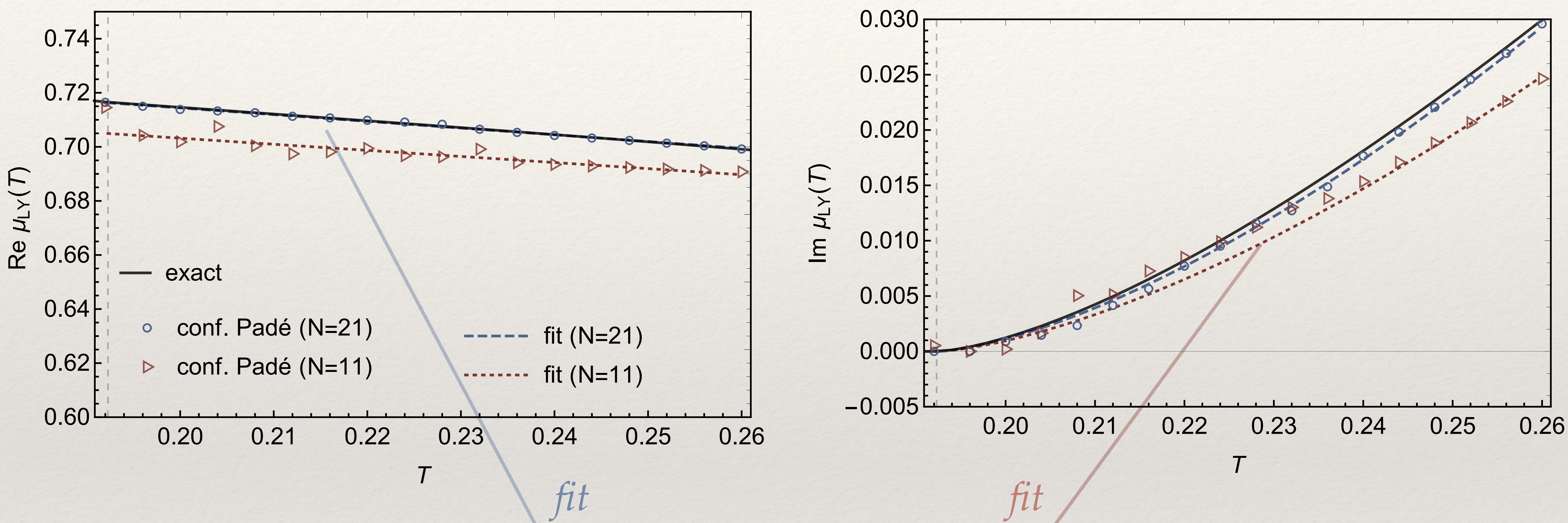
- Wuppertal-Budapest data has different results for the higher cumulants compared to HotQCD, yet Im LY seems a consistent trend with the existence of a critical point around T ~ 100 MeV

- Role of other singularities (O(4) and Roberge-Weiss) need to be understood better for a complete picture

EXTRAS

Lee-Yang trajectory

- Find $\mu_{LY}^2(T)$ from poles of the conformal-Padé (GN model)



$$\mu_{LY}(T) \approx \mu_c - \frac{h_T}{h_\mu}(T - T_c) + i w_{LY} \frac{r_\mu^{3/2}}{h_\mu}\left(\frac{r_T}{r_\mu} - \frac{h_T}{h_\mu}\right)^{3/2}(T - T_c)^{3/2} \qquad w_{LY} = \frac{2}{3\sqrt{3}}$$

- Extract $\mu_c, T_c,$ crossover slope, $\frac{h_T}{h_\mu}$, and $\frac{r_\mu^{3/2}}{h_\mu}\left(\frac{r_T}{r_\mu} - \frac{h_T}{h_\mu}\right)^{3/2}$

Ising parameters

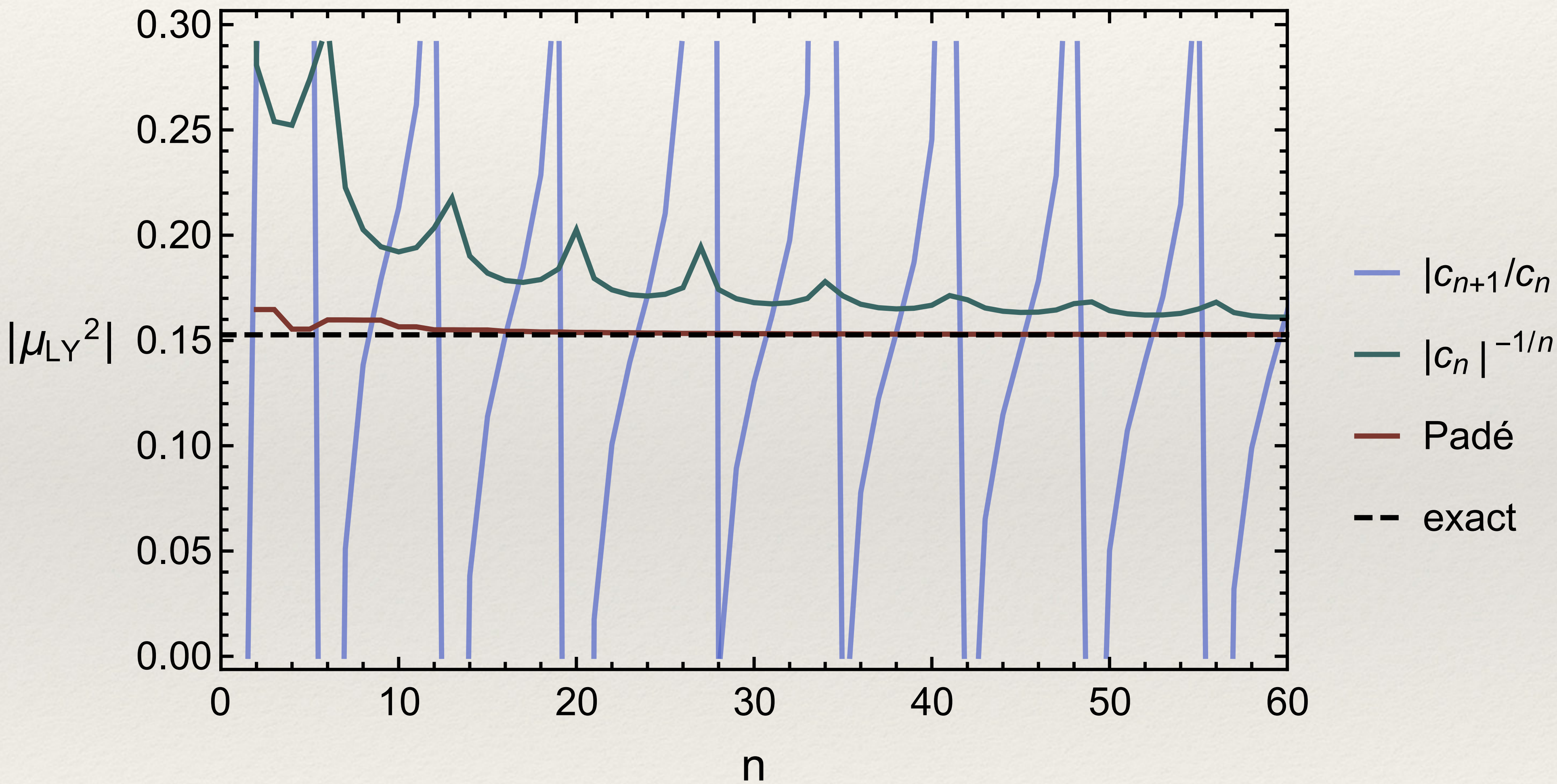
$$\mu_{LY}(T) \approx \mu_c - \frac{h_T}{h_\mu}(T - T_c) + i w_{LY} \frac{r_\mu^{3/2}}{h_\mu} \left(\frac{r_T}{r_\mu} - \frac{h_T}{h_\mu} \right)^{3/2} (T - T_c)^{3/2}$$

$$w_{LY} = \frac{2}{3\sqrt{3}}$$

	T_c	μ_c	h_T/h_μ	c
exact	0.192	0.717	0.249	4.684
conf. Padé (N=21)	0.195	0.716	0.248	4.323
conf. Padé (N=11)	0.185	0.707	0.225	3.666

When life gives you Taylor series...

Random Matrix Model

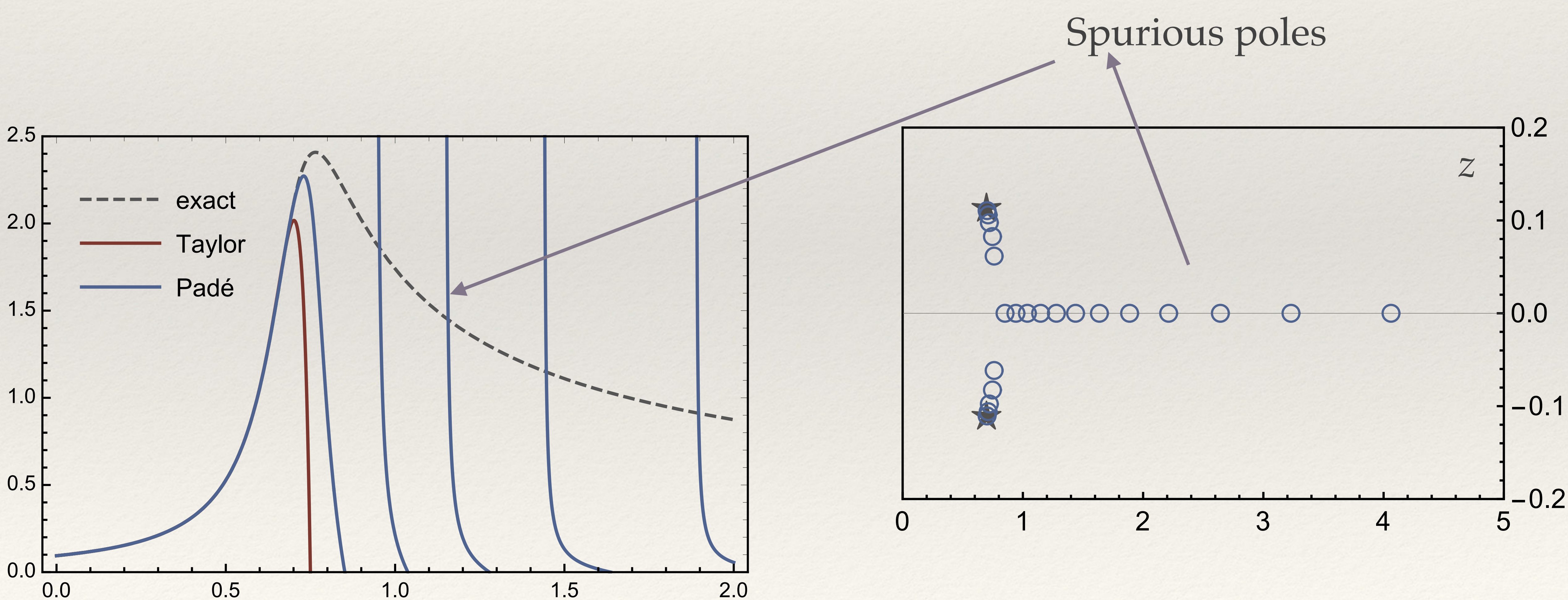


[GB, Dunne, Yin, arXiv: 2112.14269]

When life gives you Taylor series…

Spurious poles are unavoidable in Padé when there are conjugate pair of singularities…

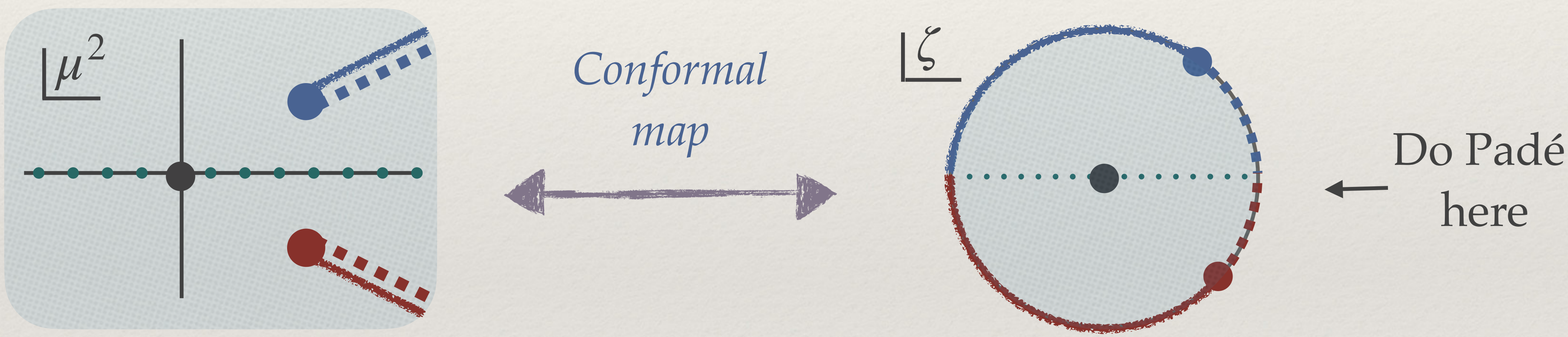
$$\text{e.g.} \quad f(z) = \frac{1}{2}\left(\frac{1}{\sqrt{z - z_c}} + \frac{1}{\sqrt{z - z_c^*}}\right)$$



Conformal Maps

Solution: Do Padé after a conformal map

- Captures the singular behavior, no unphysical poles along real axis ✓
- Significantly better approximation than Padé ✓

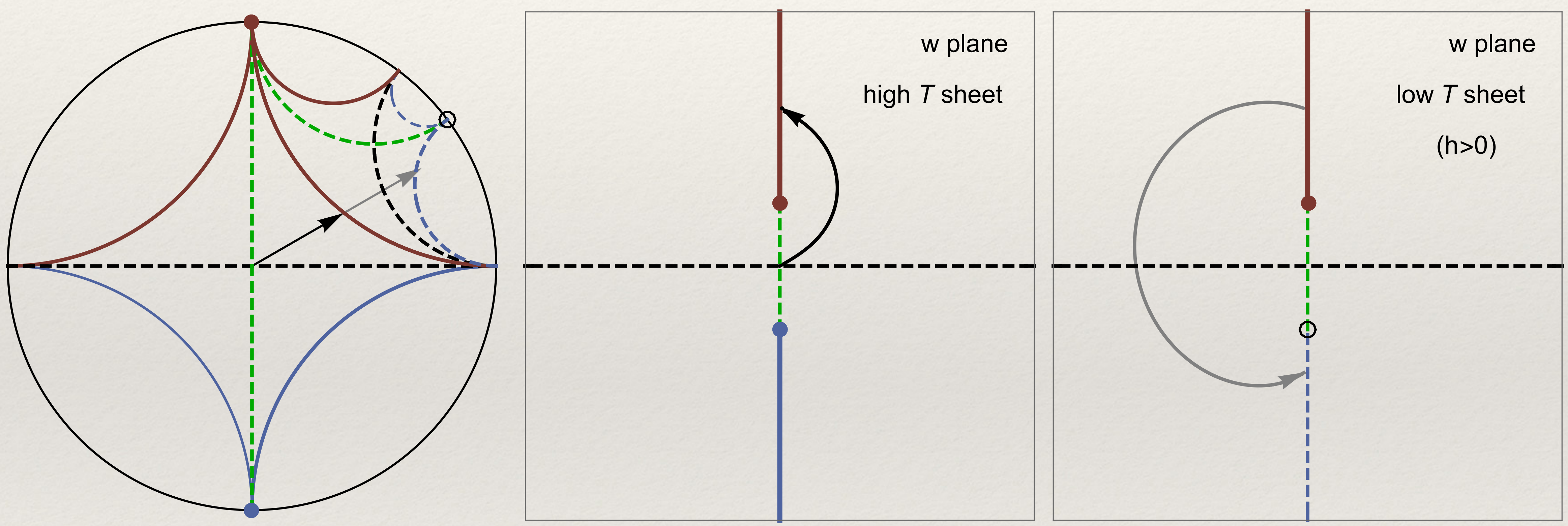


$$\phi(\zeta) = \left(\frac{\theta}{\pi}\right)^{\theta/\pi}\left(1-\frac{\theta}{\pi}\right)^{1-\theta/\pi}\frac{4\mu_{LY}^2\zeta}{(1+\zeta)^2}\left(\frac{1+\zeta}{1-\zeta}\right)^{2\theta/\pi}$$

"conformal Padé"

$$\mathrm{P}\chi(T,\phi(\zeta)) = \left.\frac{\tilde{p}_0(T)+\tilde{p}_1(T)\zeta+\ldots+\tilde{p}_{N/2}(T)\zeta^N}{\tilde{q}_0(T)+\tilde{q}_1(T)\zeta+\ldots+\tilde{q}_{N/2}(T)\zeta^N}\right|_{\zeta=\phi^{-1}(\mu^2)}$$

Uniformization Map : crossing the branch cut



Moving within unit circle (smooth) ⟷ *Jumping through sheets*

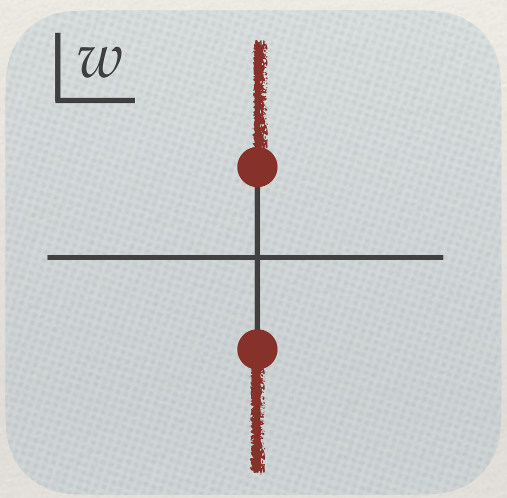
Uniformization Map : crossing the branch cut

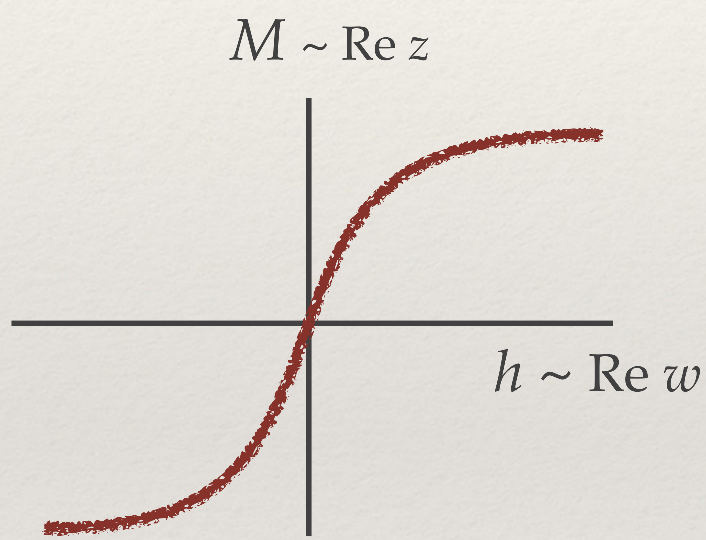
$w = hr^{-\beta\delta}$
$z = Mr^{-\beta}$

Ising model: $\quad w = F(z)$

$F(z) = z + z^3 \quad$ (mean field)

High Temperature ($T>T_c$)





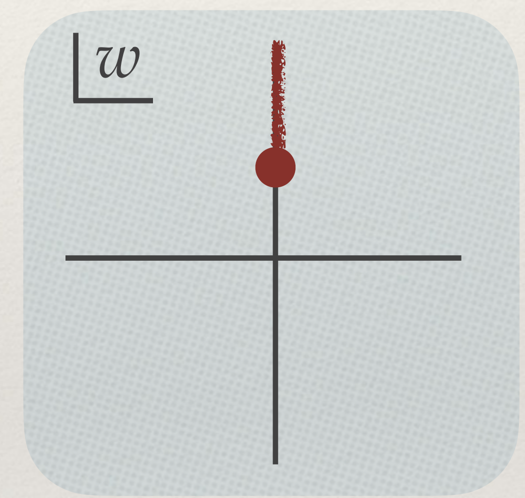
high T sheet
r>0

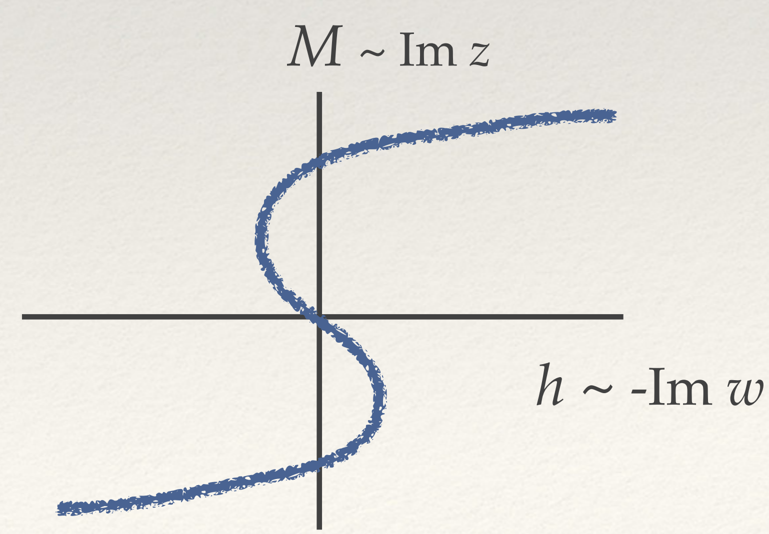
$$z(w) = w - w^3 + 3w^5 - 12w^7 + \dots$$

high T expansion

Low Temperature ($T<T_c$)

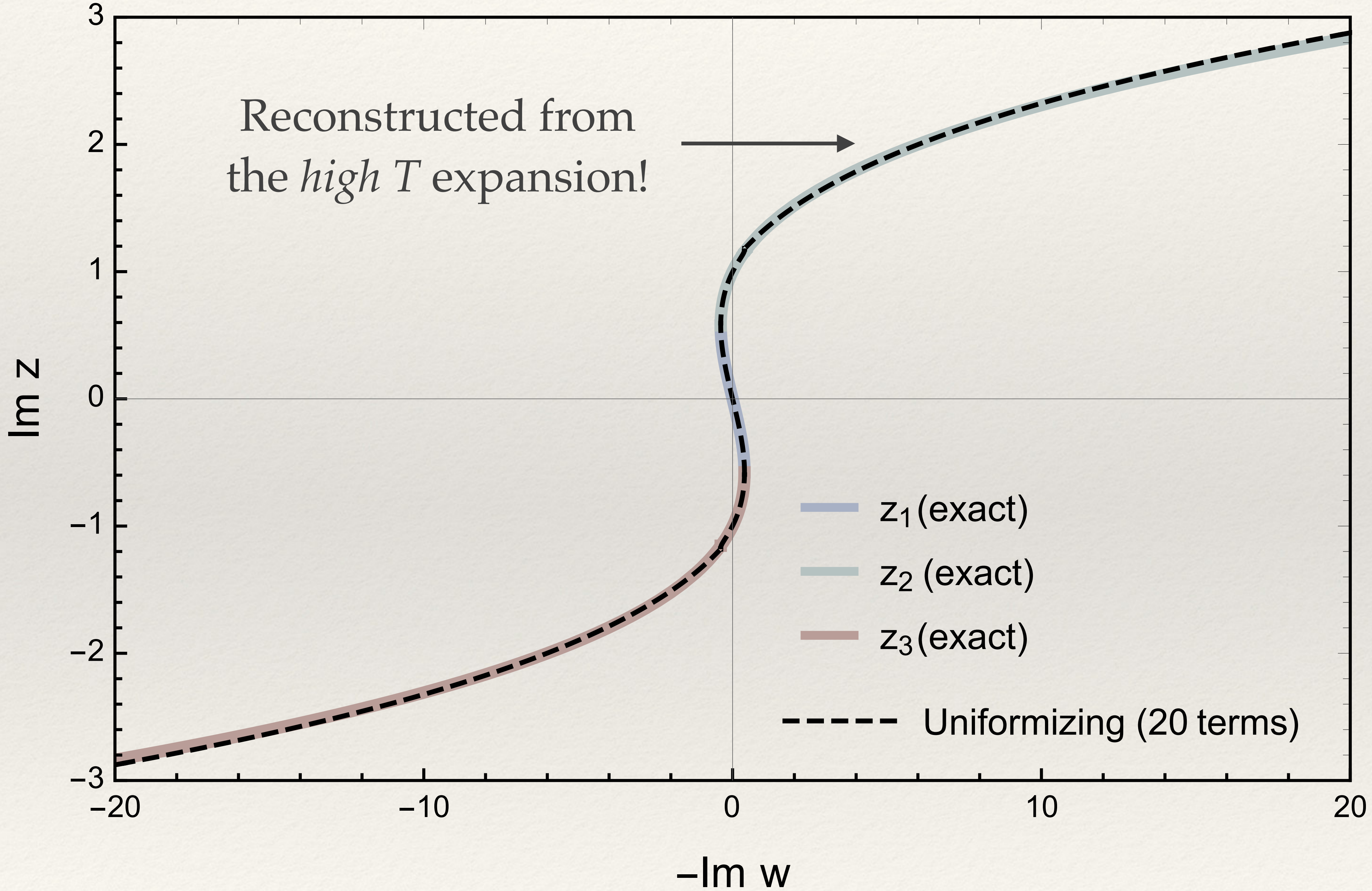
low T sheet
r<0, h>0





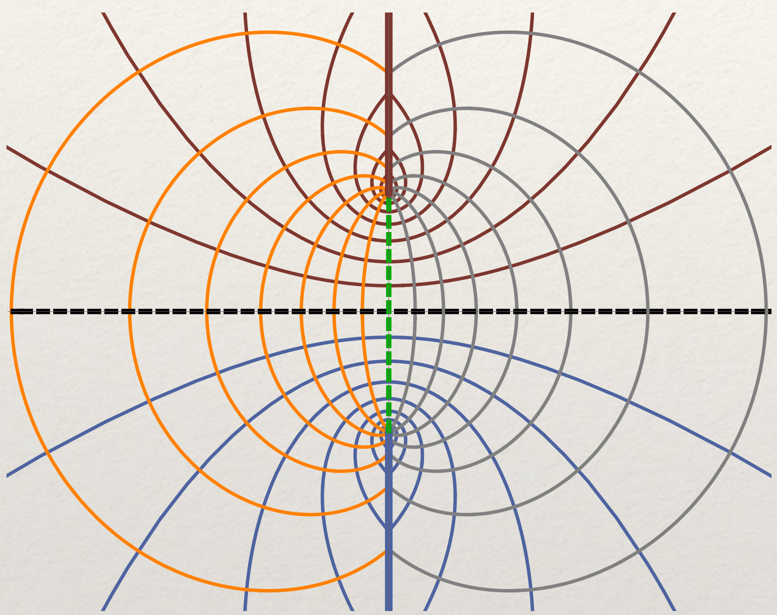
Uniformization: crossing the branch cut

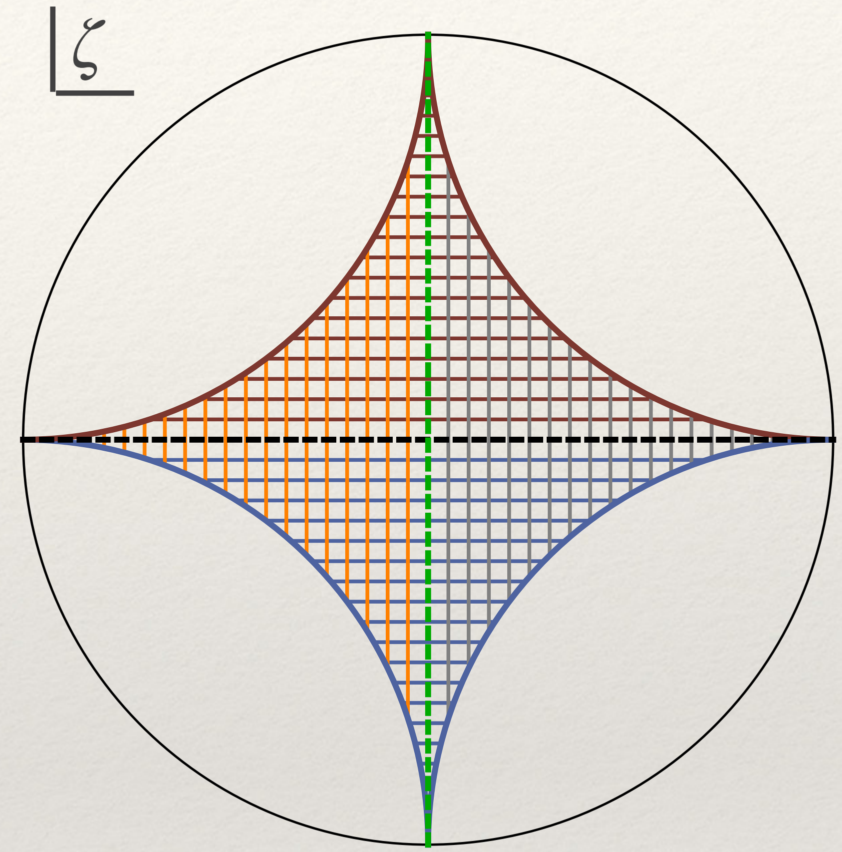
Low T



Uniformization: crossing the branch cut

w plane





high T sheet

r>0

$$w \to w(\tau) = i(-1 + 2\lambda(\tau))$$

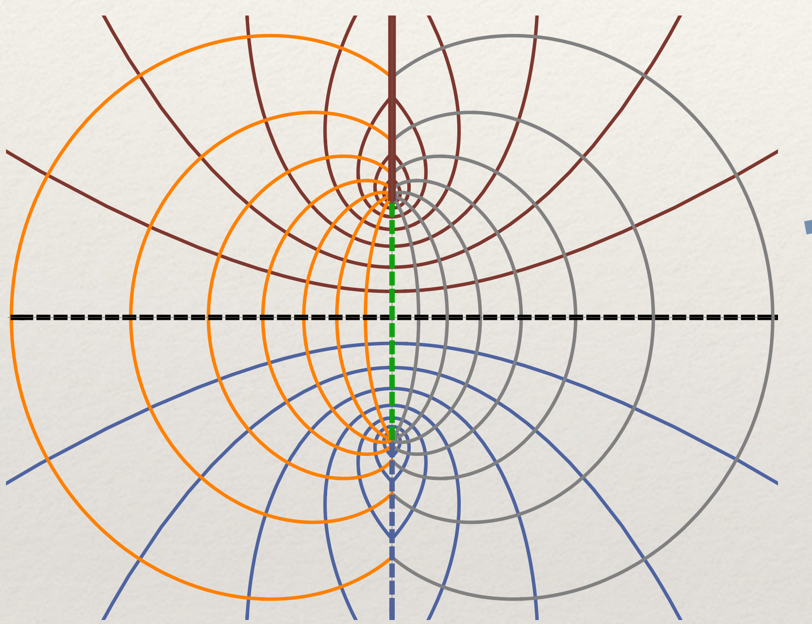
$$\tau(\zeta) = i\left(\frac{1+i\zeta}{1-i\zeta}\right)$$

$$\lambda(\tau) = \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)} \quad \text{(elliptic modular function)}$$

$$\theta_2(\tau) = \sum_{n=1}^{\infty} e^{2\pi i\tau(n+1/2)^2}, \quad \theta_3(\tau) = \sum_{n=1}^{\infty} e^{2\pi i\tau n^2}$$

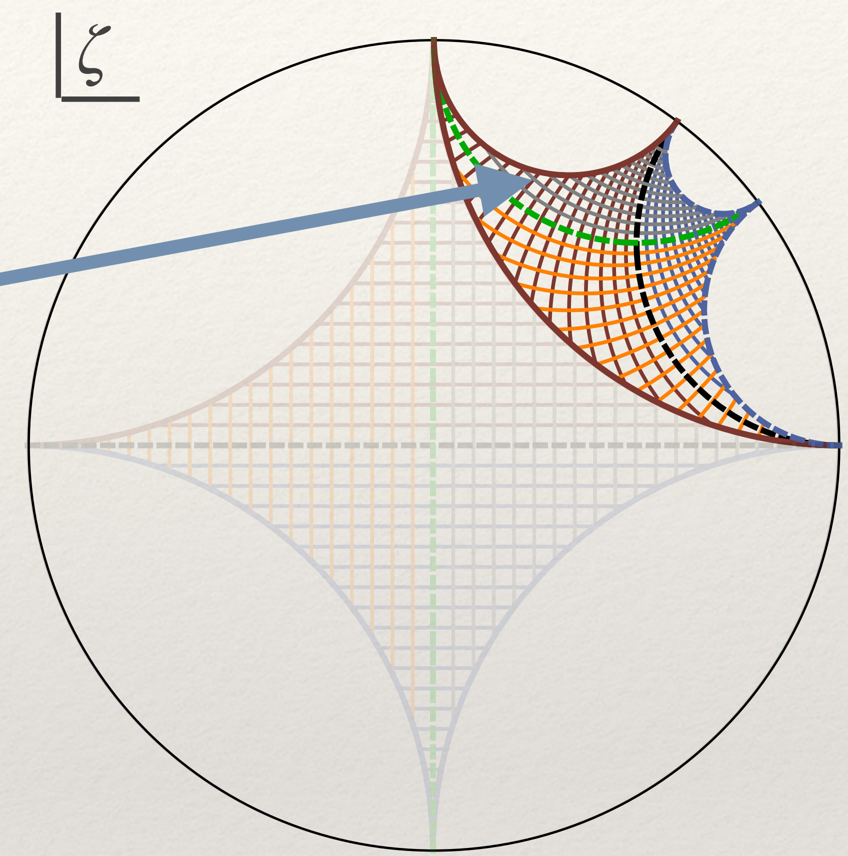
Uniformization: crossing the branch cut

w plane



low T sheet

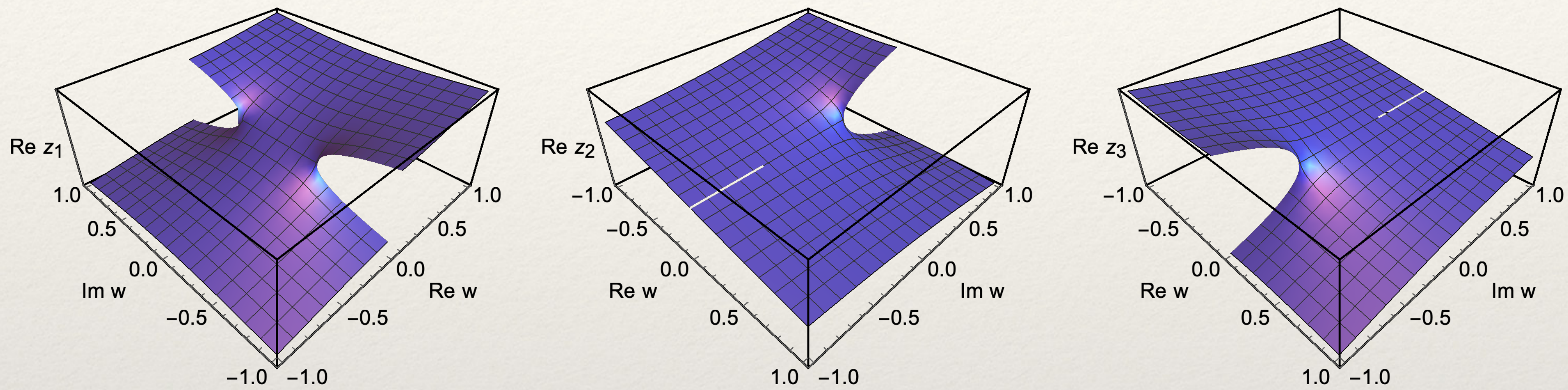
r<0



Low T sheet = Schwartz reflection of the high T sheet
(modular transformation)

Uniformization

$$w = F(z) = z + z^3 \quad \text{(mean field)}$$



$$z_1(w) = -\frac{2i}{\sqrt{3}}\left[{}_2F_1\left(\frac{1}{3}, -\frac{1}{3}, \frac{1}{2}; \frac{1}{2}(1 - iw)\right) - \text{c.c.}\right]$$

$$w(\tau) = i(-1 + 2\lambda(\tau))$$

$$z_2(w) = \frac{2i}{\sqrt{3}}\,{}_2F_1\left(\frac{1}{3}, -\frac{1}{3}, \frac{1}{2}; \frac{1}{2}(1 - iw)\right)$$

$z(\tau)$: single valued

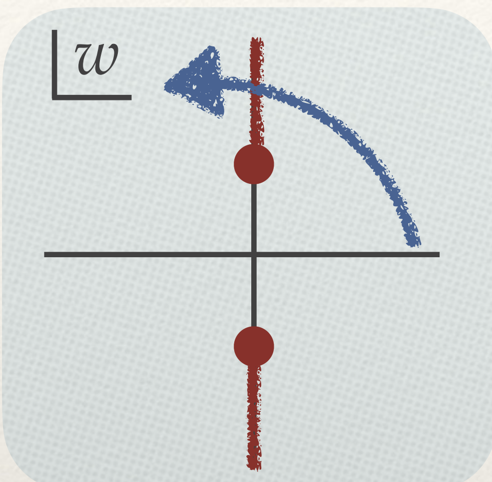
"uniformization"

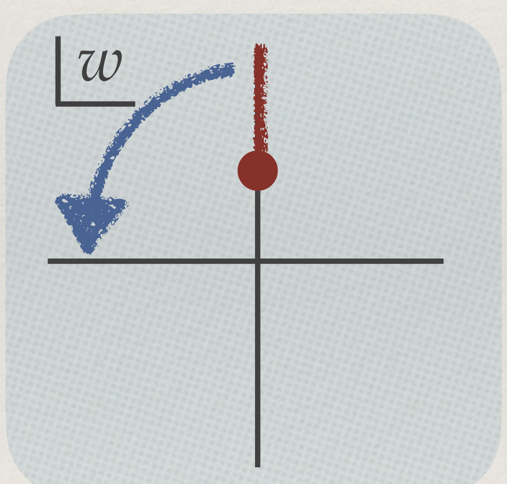
$$\lambda(\tau) = \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)} \quad \theta_2(\tau) = \sum_{n=-\infty}^{\infty} e^{2\pi i\tau(n+1/2)^2}, \quad \theta_3(\tau) = \sum_{n=-\infty}^{\infty} e^{2\pi i\tau n^2}$$

[Bateman, Higher Transcendal Functions I]

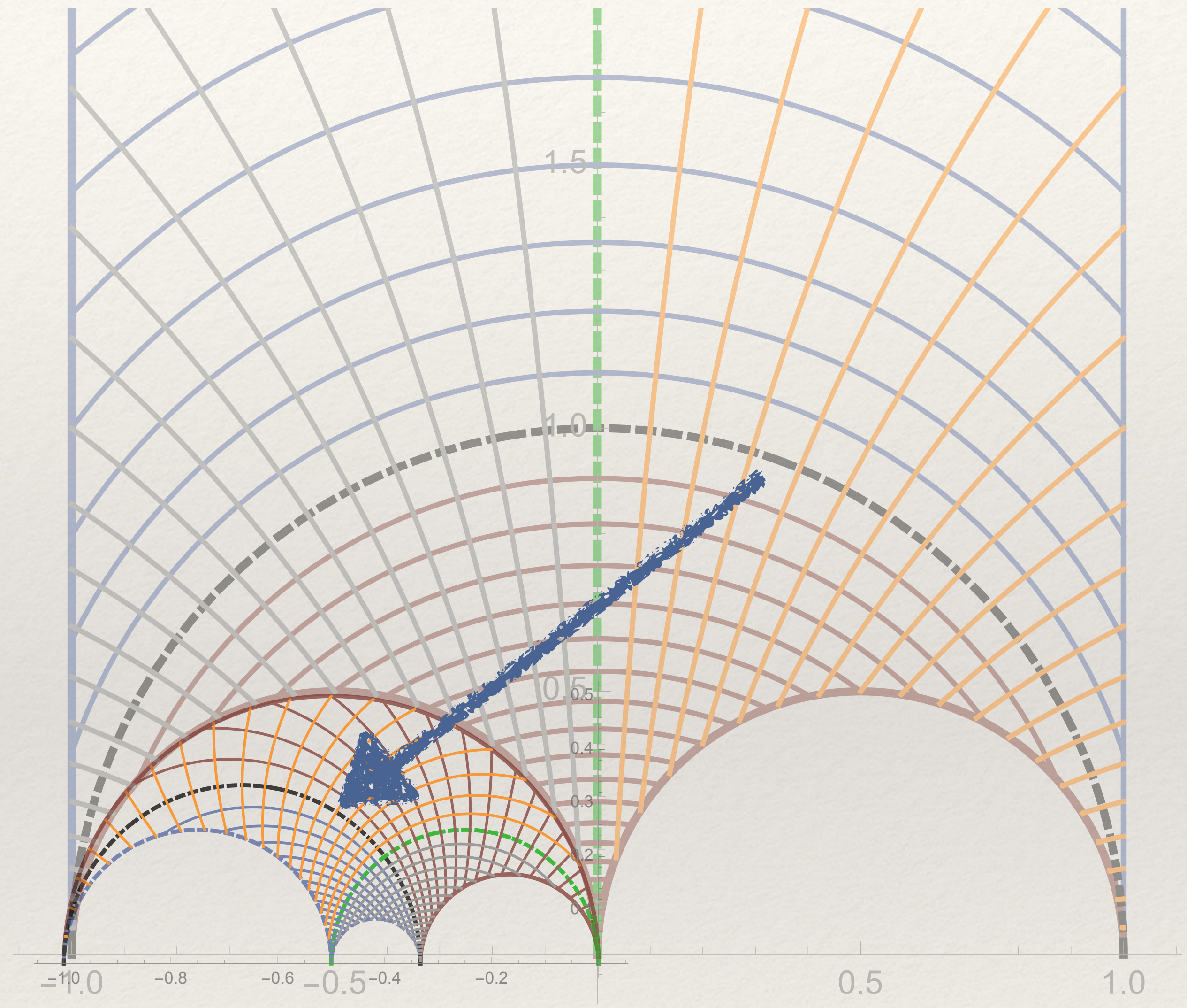
* $w \to 2/(3\sqrt{3})w$

Uniformization





Jumping sheets
in w plane

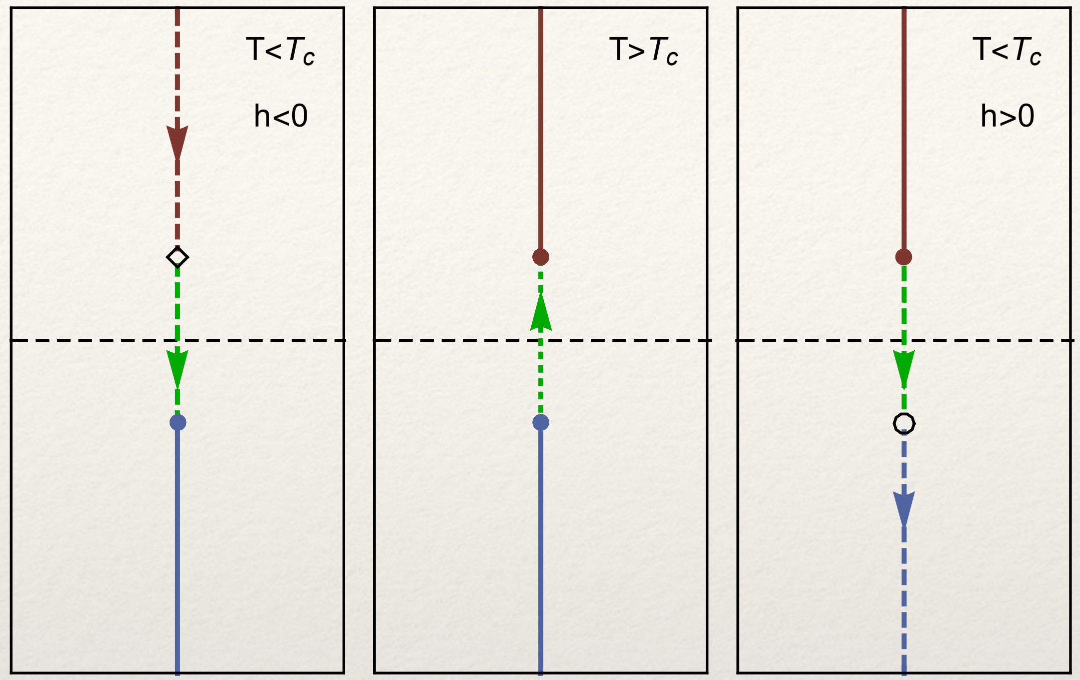


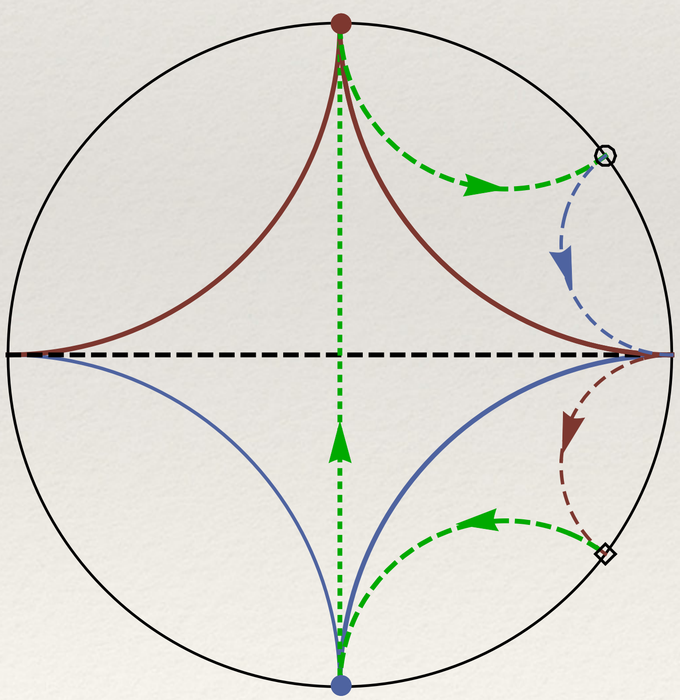
Smooth in
τ plane

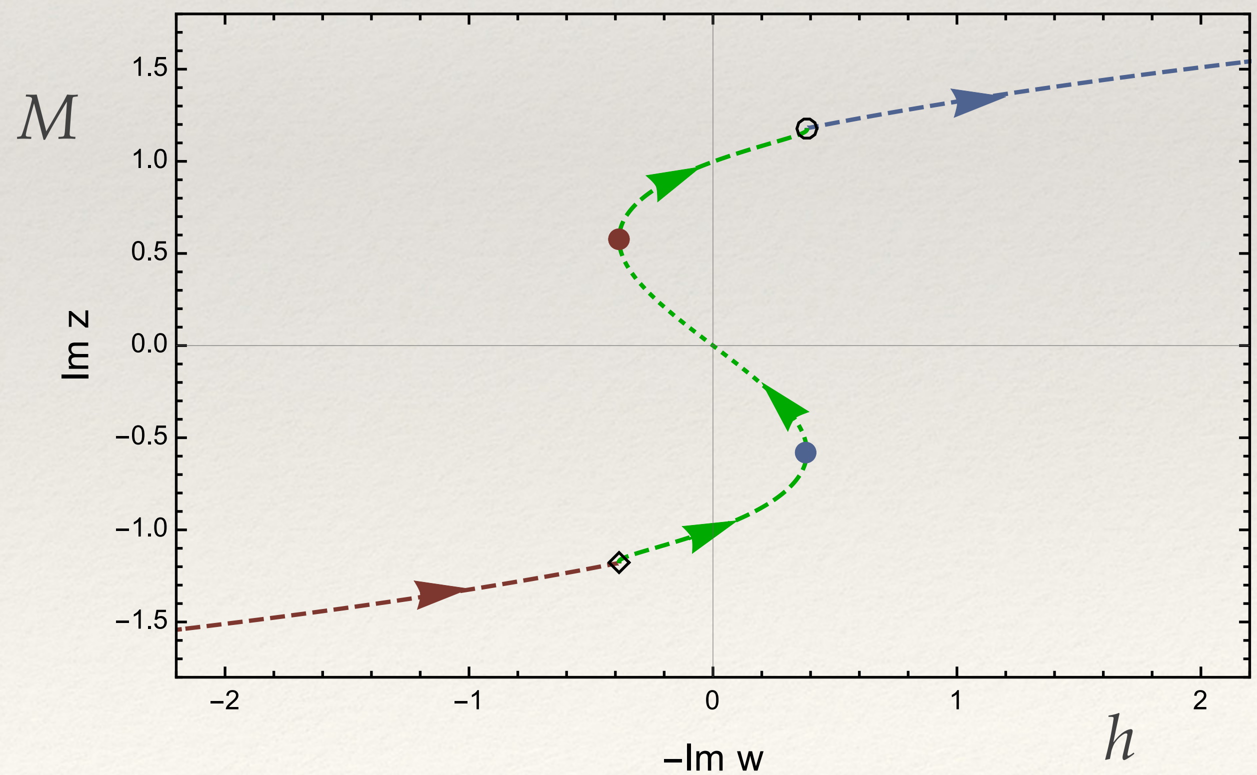
Interactive realization:
https://people.math.osu.edu/costin.9/classes.html

*

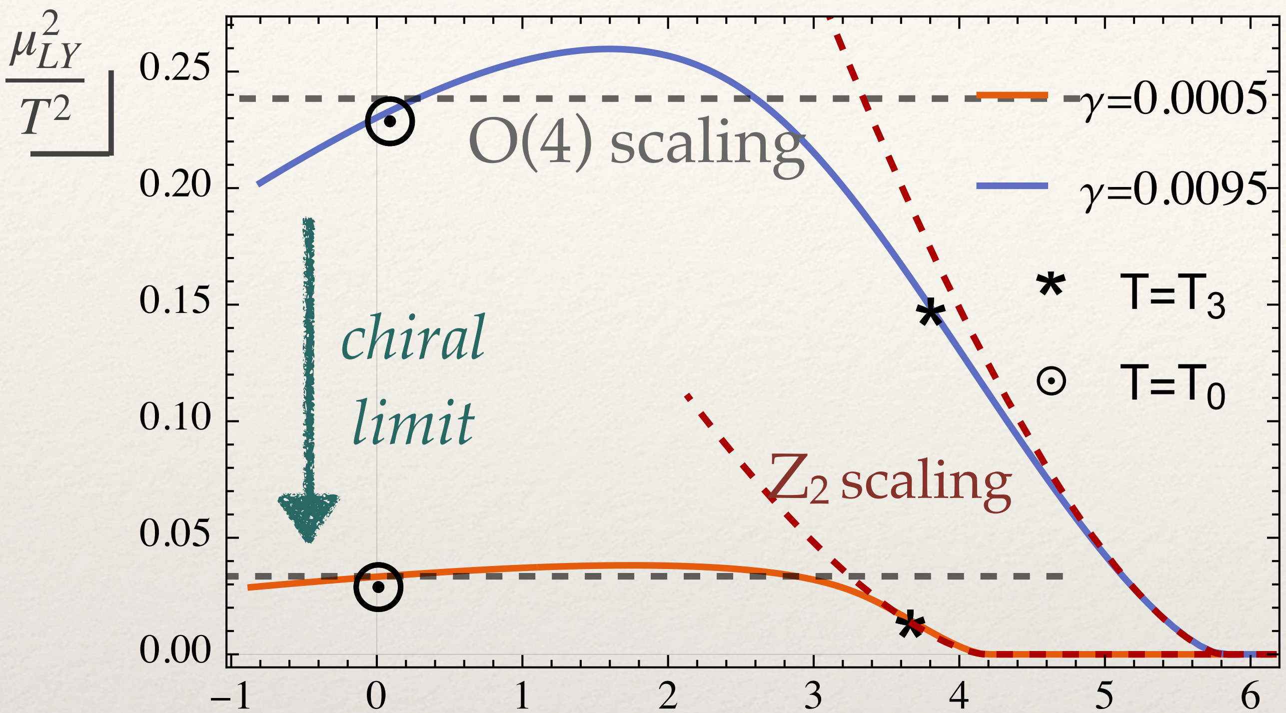
Uniformization: higher Riemann sheets



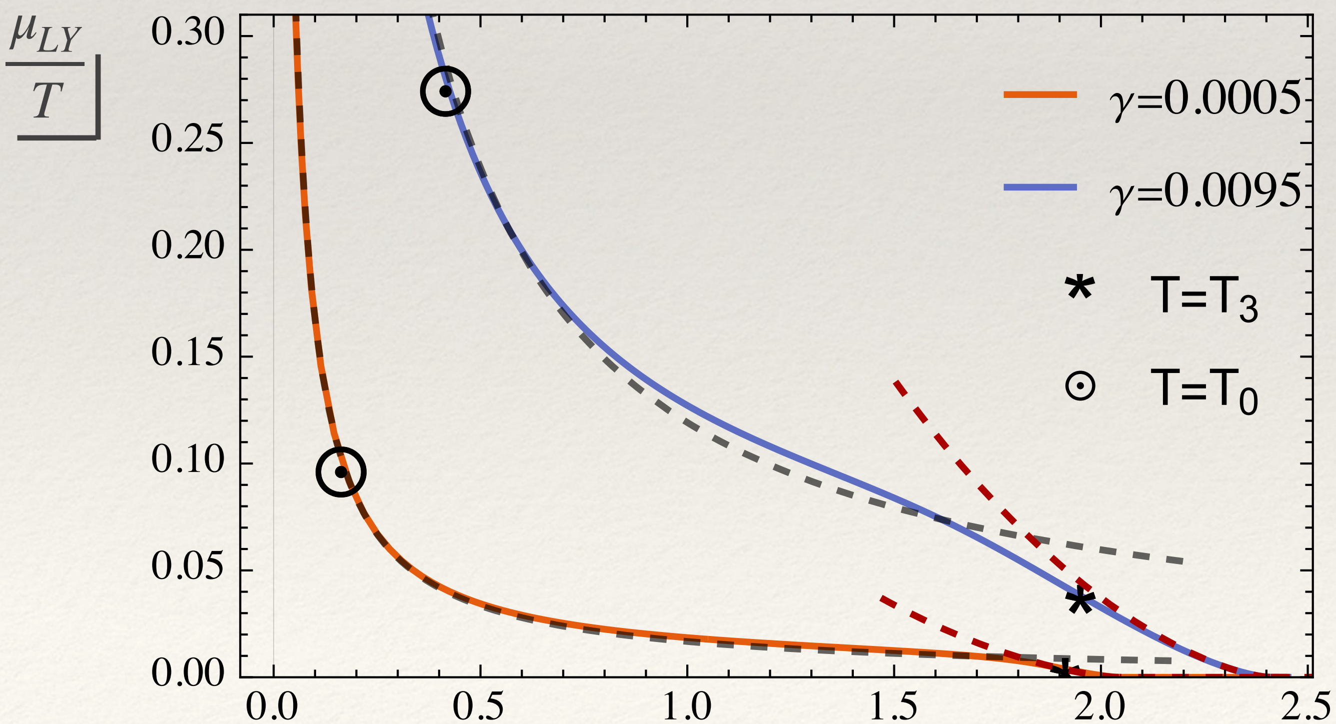




A parade of singularities



Gross-Neveu



"O(4) scaling"

$$r \propto \frac{T - T_0}{T_0} - \kappa_2 \frac{\mu^2}{T^2}, \quad h \propto m_l/m_s$$

[Ejiri et al 0909.5122]

$$hr^{-\beta\delta} = \pm\, i w_{LY}$$

$$\frac{\mu_{LY}^2}{T^2} = a + b(T - T_0) \pm i(\text{constant})$$