Helicity and orbital angular momentum at Small x

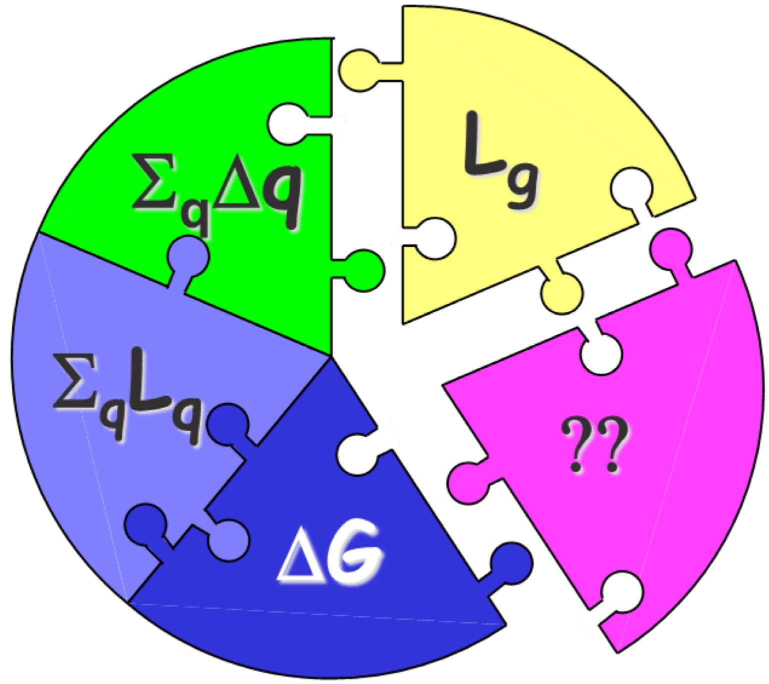
Yoshitaka Hatta & Yuri Kovchegov

Proton Spin

Proton Helicity Sum Rule

- Helicity sum rule (Jaffe-Manohar form):

$$\frac{1}{2} = S_q + L_q + S_g + L_g$$



with the net quark and gluon spin

$$S_q(Q^2) = \frac{1}{2} \int_0^1 dx \, \Delta\Sigma(x, Q^2) \qquad S_g(Q^2) = \int_0^1 dx \, \Delta G(x, Q^2)$$

- L_q and L_g are the quark and gluon orbital angular momenta

Proton Spin Puzzle

$$S_q(Q^2) = \frac{1}{2} \int_0^1 dx \, \Delta\Sigma(x, Q^2)$$

- The spin puzzle began when the EMC collaboration measured the proton g_1 structure function ca 1988. Their data resulted in

$$\Delta\Sigma \approx 0.1 \div 0.2$$

- It appeared quarks do not carry all of the proton spin (which would have corresponded to $\Delta\Sigma = 1$).

- Missing spin can be

$$\frac{1}{2} = S_q + L_q + S_g + L_g$$

  - Carried by gluons
  - In the orbital angular momenta of quarks and gluons
  - At small x:

$$S_q(Q^2) = \frac{1}{2} \int_0^1 dx \, \Delta\Sigma(x, Q^2) \qquad S_g(Q^2) = \int_0^1 dx \, \Delta G(x, Q^2)$$

Can't integrate down to zero, use x_{min} instead!

  - Or all of the above!

Polarized PDF global analysis

Table by E. Nocera

Recent determinations of polarised PDFs

	DSSV	NNPDF	JAM
DIS	☑	☑	☑
SIDIS	☑	☒	☑
pp	☑ (jets, π^0)	☑ (jets, $W^{\pm}$)	☒
statistical treatment	Lagr. mult. $\Delta\chi^2/\chi^2 = 2\%$ Monte Carlo	Monte Carlo	Monte Carlo
parametrization	polynomial (23 pars)	neural network (259 pars)	polynomial (10 pars)
features	global fit	minimally biased fit	large-x effects
latest updates	DSSV08 PRD 80 (2009) 034030 DSSV14 PRL 113 (2014) 012001	NNPDFpol1.0 NPB 874 (2013) 36 NNPDFpol1.1 NPB 887 (2014) 276	JAM15 PRD 93 (2016) 074005 JAM17 PRL 119 (2017) 132001

$$\int_{0.05}^{1} dx\,\Delta g(x,Q^2=10\,\mathrm{GeV}^2) = 0.20^{+.06}_{-.07} \qquad \text{DSSV++}$$

$$\int_{0.05}^{0.2} dx\,\Delta g(x,Q^2=10\,\mathrm{GeV}^2) = 0.17\pm 0.06 \qquad \text{NNPDFpol1.1}$$

$$\int_{0.001}^{0.8} dx\,\Delta g(x,Q^2 = 1\,\mathrm{GeV}^2) = 0.5\pm 0.4 \qquad \text{JAM15}$$

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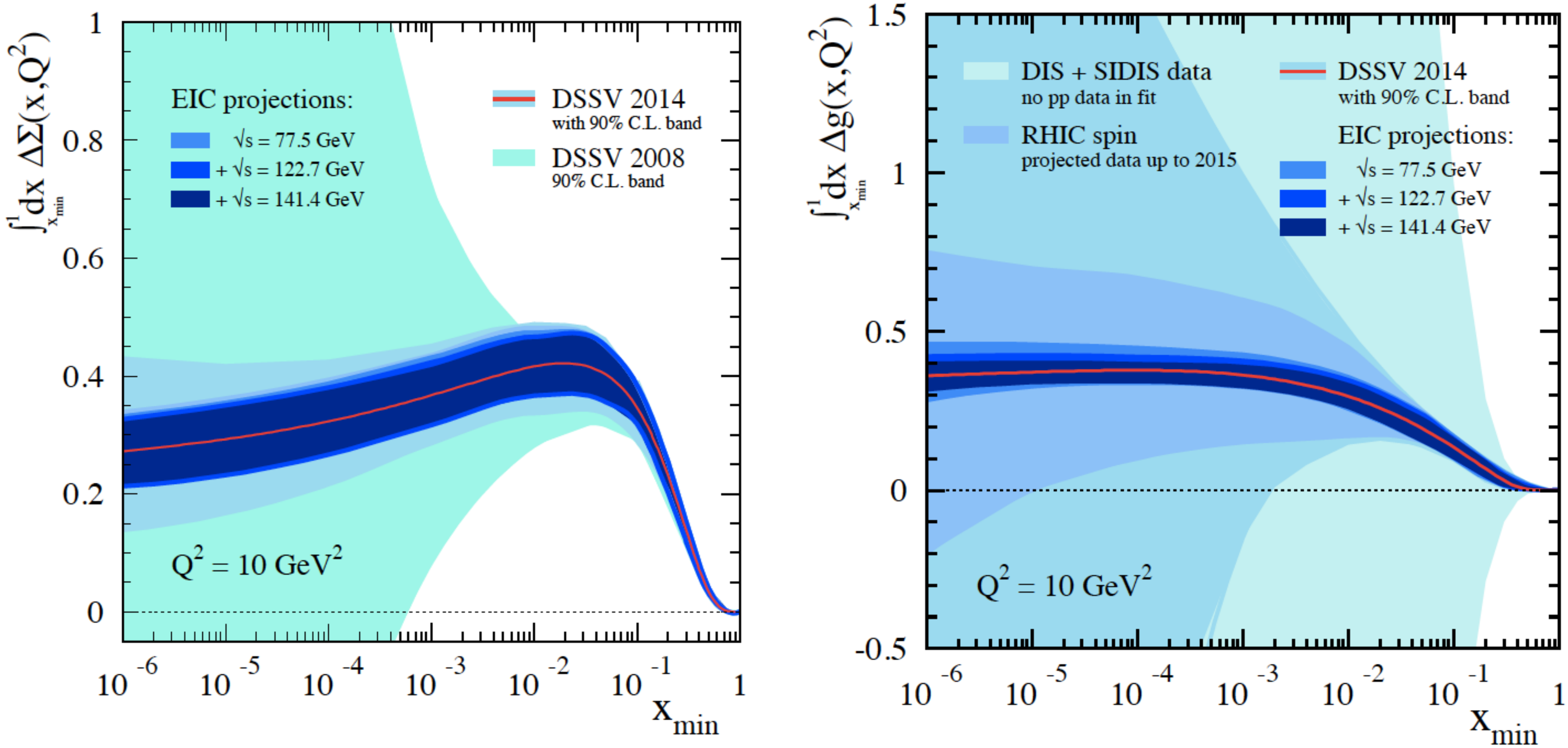
How much spin is at small x?



- E. Aschenaur et al, arXiv:1509.06489 **[hep-ph]**, (DSSV = de Florian, Sassot, Stratmann, Vogelsang, helicity PDF parametrization)
- Uncertainties are very large at small x! (EIC may reduce them.)

Helicity PDFs at Small-x

- Theoretical calculations by Bartels, Ermolaev and Ryskin (BER, 1996) and by YK, Pitonyak and Sievert (2015-17, KPS).
- KPS results (large-N_C):

$$\Delta q(x, Q^2) \sim \left(\frac{1}{x}\right)^{\alpha_h^q} \quad \text{with} \quad \alpha_h^q = \frac{4}{\sqrt{3}} \sqrt{\frac{\alpha_s\, N_c}{2\pi}} \approx 2.31 \sqrt{\frac{\alpha_s\, N_c}{2\pi}}$$

$$\Delta G(x, Q^2) \sim \left(\frac{1}{x}\right)^{\alpha_h^G} \quad \text{with} \quad \alpha_h^G = \frac{13}{4\sqrt{3}} \sqrt{\frac{\alpha_s\, N_c}{2\pi}} \approx 1.88 \sqrt{\frac{\alpha_s\, N_c}{2\pi}}$$

- BER results at large-N_C (for both quark and gluon helicity distributions):

$$\alpha_h^{BER} = \sqrt{\frac{17+\sqrt{97}}{2}} \sqrt{\frac{\alpha_s N_c}{2\pi}} \approx 3.66 \sqrt{\frac{\alpha_s N_c}{2\pi}}$$

For finite N_C and N_f=4 BER have 3.66 → 3.45.

Helicity PDFs at Small-x

$$PDF(x) \sim \left(\frac{1}{x}\right)^{\alpha}$$

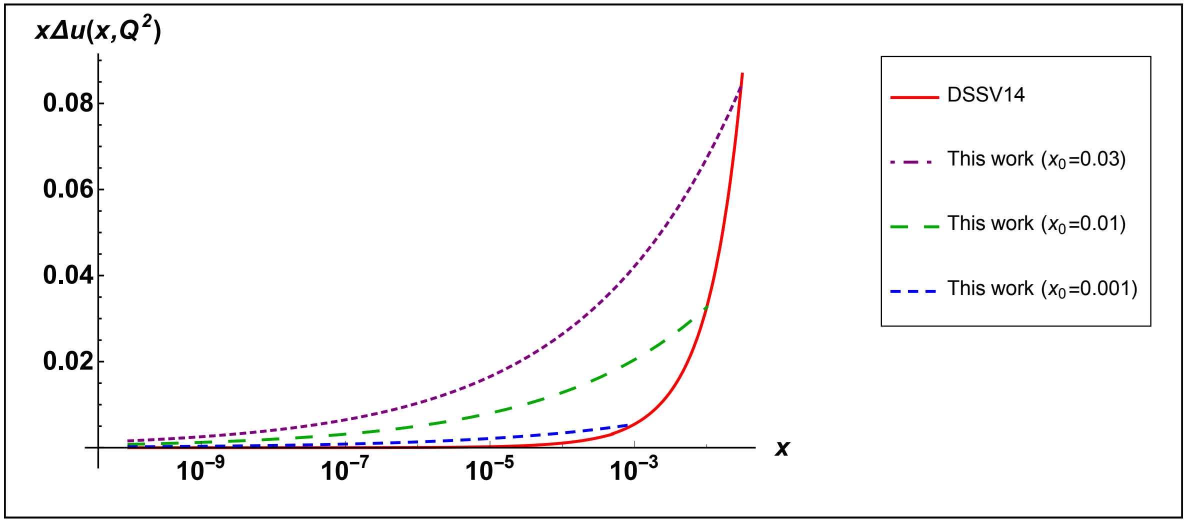
- The summary of the existing powers of x is

Observable	Evolution	Intercept	$Q^2 = 3$ GeV^2 $\alpha_s = 0.343$	$Q^2 = 10$ GeV^2 $\alpha_s = 0.249$	$Q^2 = 87$ GeV^2 $\alpha_s = 0.18$
Unpolarized flavor singlet structure function F_2	LO BFKL Pomeron	$1 + \frac{\alpha_s N_c}{\pi} 4 \ln 2$	1.908	1.659	1.477
Unpolarized flavor non-singlet structure function F_2	Reggeon	$\sqrt{\frac{2\,\alpha_s C_F}{\pi}}$	0.540	0.460	0.391
Flavor singlet structure function g_1^S	us (Pure Glue)	$2.31 \sqrt{\frac{\alpha_s N_c}{2\pi}}$	0.936	0.797	0.678
	BER (Pure Glue)	$3.66 \sqrt{\frac{\alpha_s N_c}{2\pi}}$	1.481	1.262	1.073
	BER ($N_f = 4$)	$3.45 \sqrt{\frac{\alpha_s N_c}{2\pi}}$	1.400	1.190	1.011
Flavor non-singlet structure function g_1^{NS}	BER and us (large-N_c)	[illegible]	0.572	0.488	0.415

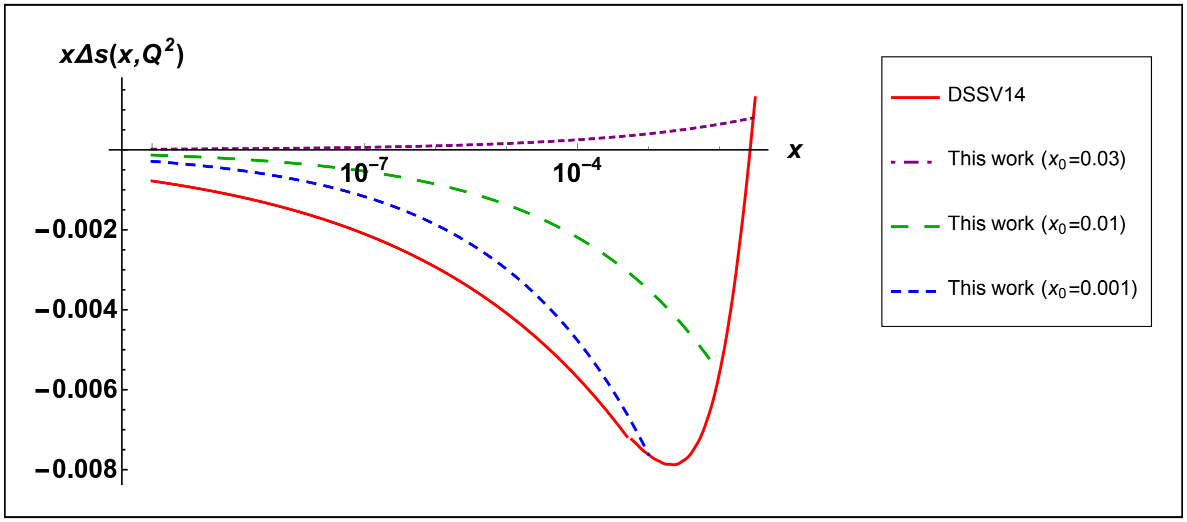
- Similar disagreement exists for OAM distributions at small x.
- However, even the smaller KPS powers lead to potentially important contributions to the proton spin coming from small x. (next)

Impact on proton spin

- We have attached a $\Delta\tilde{\Sigma}(x, Q^2) = N\, x^{-\alpha_h}$ curve to the existing hPDF's fits at some ad hoc small value of x labeled x_0 :

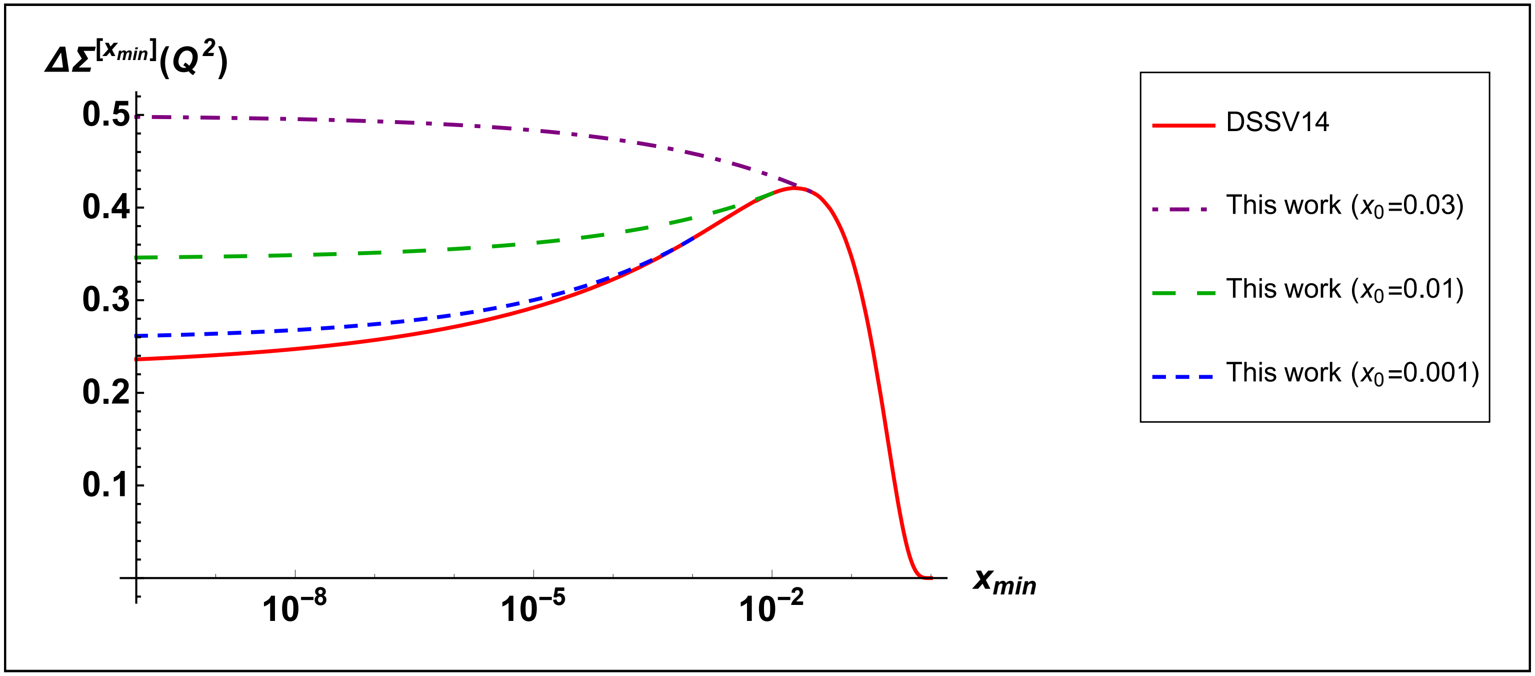


"ballpark" phenomenology



Impact on proton spin

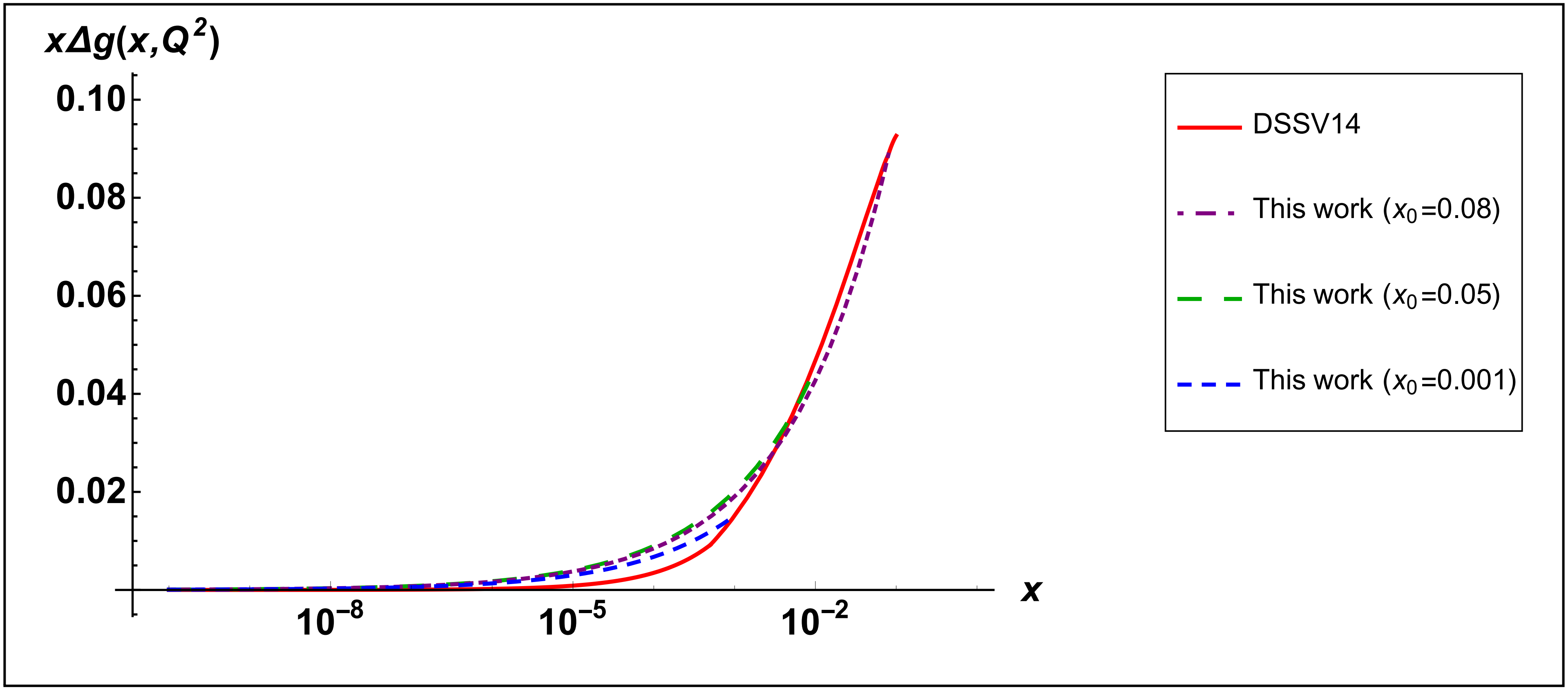
- Defining $\Delta\Sigma^{[x_{min}]}(Q^2) \equiv \int_{x_{min}}^{1} dx\, \Delta\Sigma(x, Q^2)$ we plot it for x_0=0.03, 0.01, 0.001:



- We observe a moderate to significant enhancement of quark spin.
- More detailed phenomenology is needed in the future.

Impact of our ΔG on the proton spin

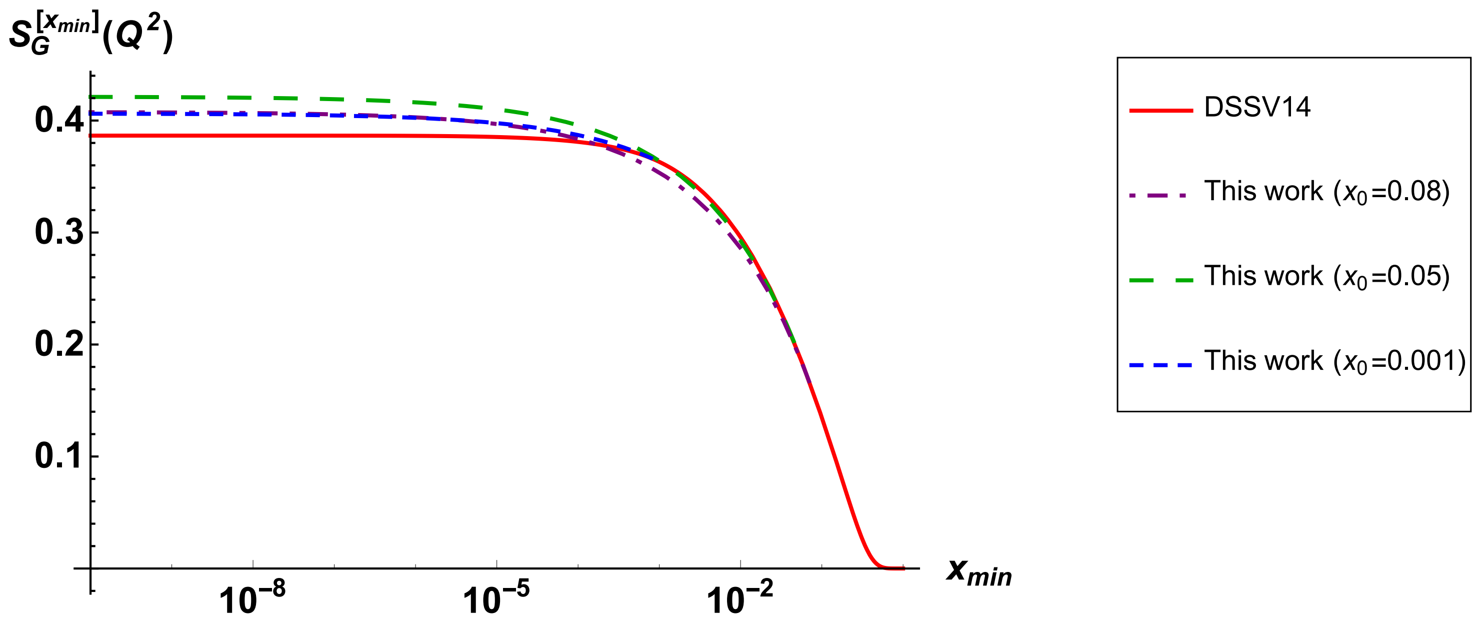
- We have attached a $\Delta\tilde{G}(x,Q^2) = N\, x^{-\alpha_h^G}$ curve to the existing hPDF's fits at some ad hoc small value of x labeled x_0 :



"ballpark"
phenomenology

Impact of our ΔG on the proton spin

- Defining $S_G^{[x_{min}]}(Q^2) \equiv \int_{x_{min}}^{1} dx\, \Delta G(x,Q^2)$ we plot it for x_0=0.08, 0.05, 0.001:

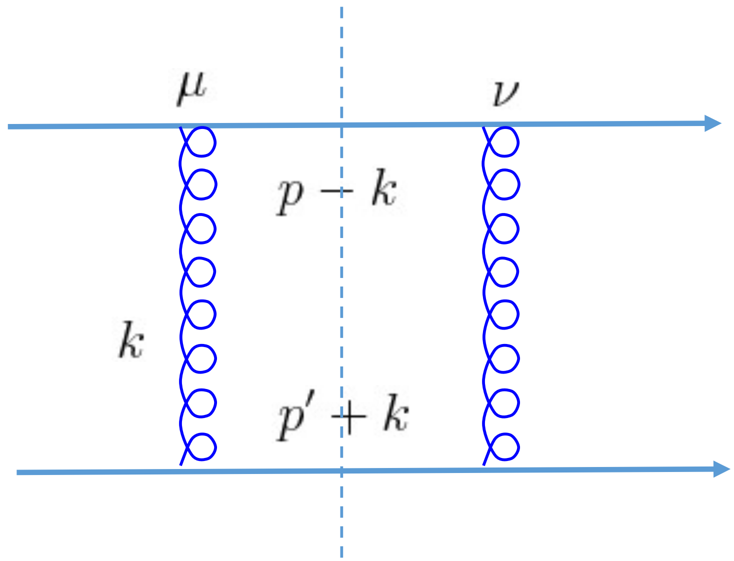


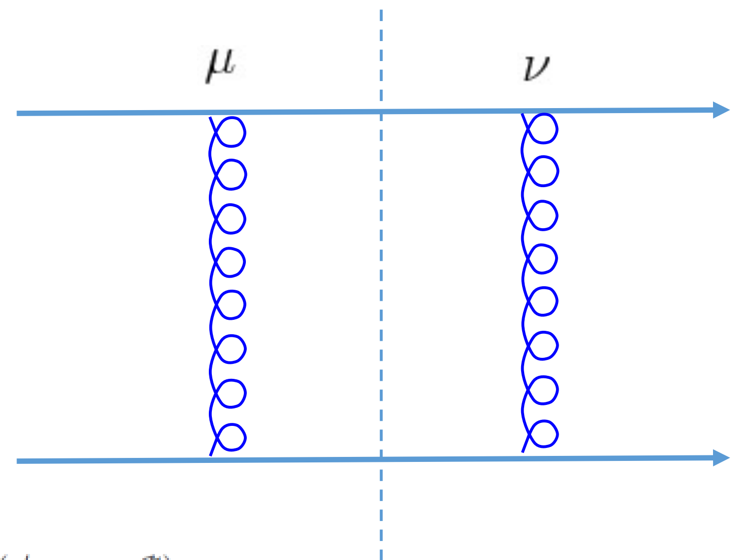
- We observe a moderate enhancement of gluon spin.
- More detailed phenomenology is needed in the future.

BER approach

Unpolarized

Polarized





$$u(PS)\bar{u}(PS) = \frac{1}{2}(\not{p} - \gamma_5 \not{S})$$

$$\mathrm{Tr}\not{p}\gamma^\mu(\not{p} - \not{k})\gamma^\nu \approx 8p^\mu p^\nu$$

$$\mathrm{Tr}\gamma_5 \not{S}\gamma^\mu(\not{p} - \not{k})\gamma^\nu \approx -4i\epsilon^{-\mu i\nu} S^+ k_i$$

$$g^4 \frac{(p\cdot p')^2}{(k^2)^2} \sim \alpha_s^2 \frac{s^2}{k_\perp^4}$$

$$g^4 \frac{p\cdot p' k_\perp^2}{(k^2)^2} \sim \alpha_s^2 \frac{s}{k_\perp^2}$$

Neglect k in the numerator
→ Eikonal approximation

Either μ or ν is transverse (sub-eikonal)
$d^2k_\perp$ integral logarithmic

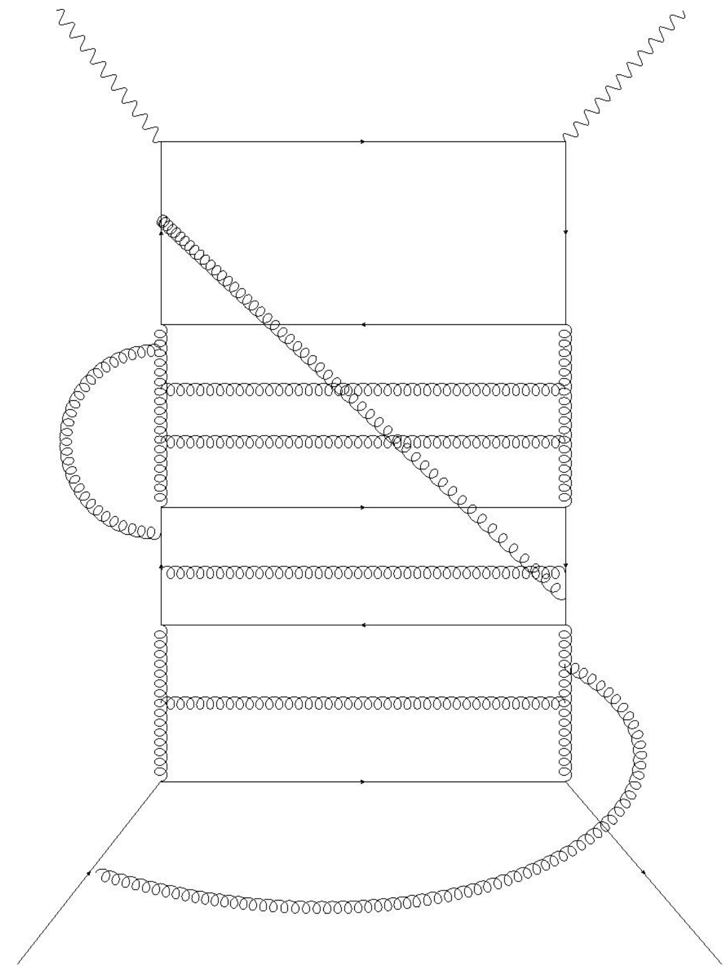
Double logarithmic approximation

All-order resummation of small-x double logarithms $(\alpha_s \ln^2 1/x)^n$
for helicity distributions

Unlike BFKL, we need to include quark ladders

Unlike BFKL, we need to include
non-ladder, `Bremsstrahlung' gluons

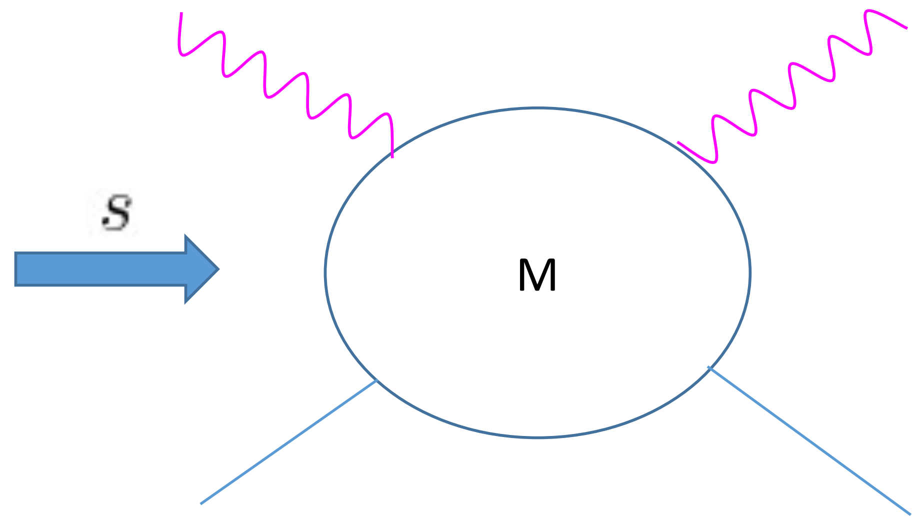
Resummation very hard, but can be done!



Infrared Evolution Equation

Kirshner, Lipatov (1983)

$$M = M(s/\mu^2, Q^2/\mu^2)$$



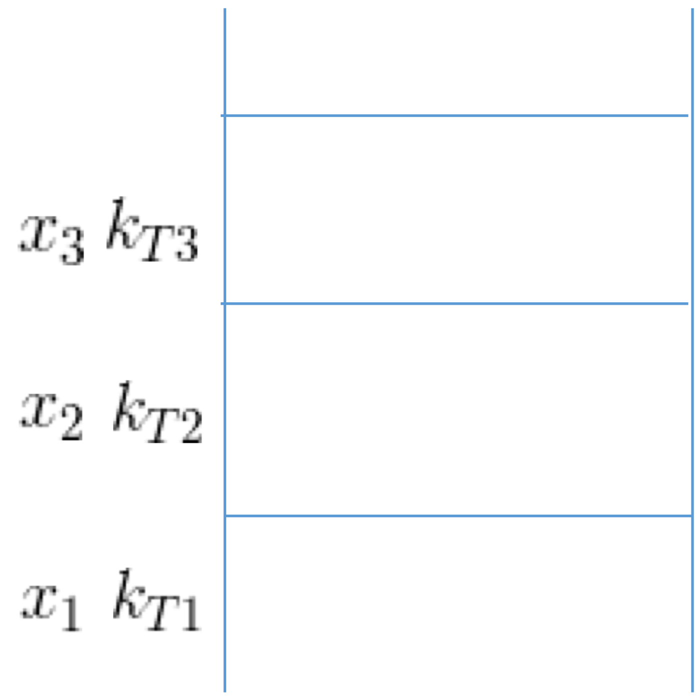
Double logs $(\ln s/Q^2)^{2n}$ come from both ladder and nonladder diagrams.

They can be organized by keeping track of the dependence on the IR cutoff μ^2

First, ladder diagrams. Double logs come from the lifetime ordering region

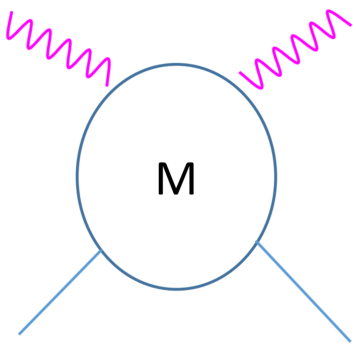
$$x_1 \gg x_2 \gg x_3 \gg \cdots$$

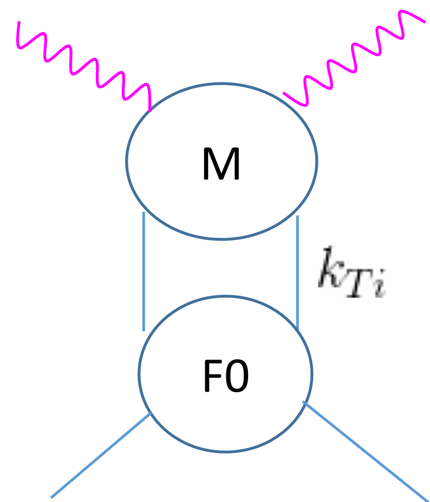
$$\frac{x_1}{k_{T1}^2} \gg \frac{x_2}{k_{T2}^2} \gg \frac{x_3}{k_{T3}^2} \gg \cdots$$

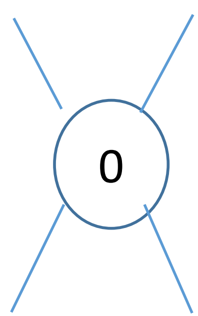


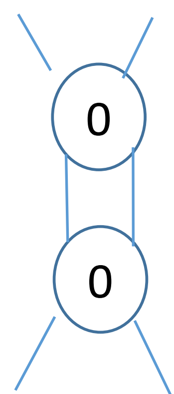
The condition $\frac{x_1}{k_{T1}^2} \gg \frac{x_2}{k_{T2}^2} \gg \frac{x_3}{k_{T3}^2} \gg \cdots$ does not mean transverse momenta are ordered.

k_{Ti} with any i can be the softest momentum $k_{Tj\neq i} \gg k_{Ti} \gg \mu$

$\frac{\partial}{\partial \ln \mu^2}$ 

$\sim \frac{\partial}{\partial \ln \mu^2} \int_{\mu^2} \frac{dk_{Ti}^2}{k_{Ti}^2} \int \frac{dx_i}{x_i}$ 

$\frac{\partial}{\partial \ln \mu^2}$ 

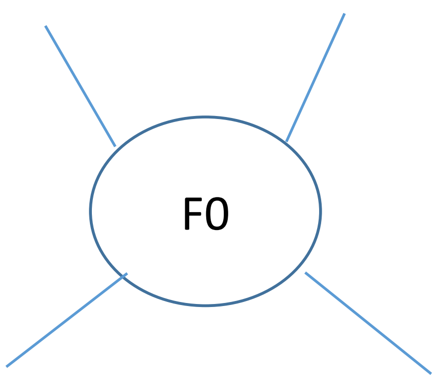
$\sim \frac{\partial}{\partial \ln \mu^2}$ 

Next, nonladder diagrams

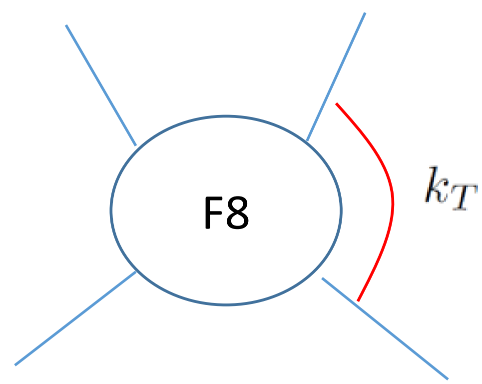
The softest gluon can be a nonladder gluon (`Bremsstrahlung gluon') attaching to the sides of the ladder.

Double logs arise only when this gluon stretches maximally, connecting the external legs. ← Gribov's bremsstrahlung theorem

$$\frac{\partial}{\partial \ln \mu^2}$$



$$\sim \frac{\partial}{\partial \ln \mu^2} \int_{\mu^2} \frac{dk_T^2}{k_T^2}$$



InfraRed Evolution Equation (IREE) for polarized structure function

Bartels, Ermolaev, Ryskin (1996)

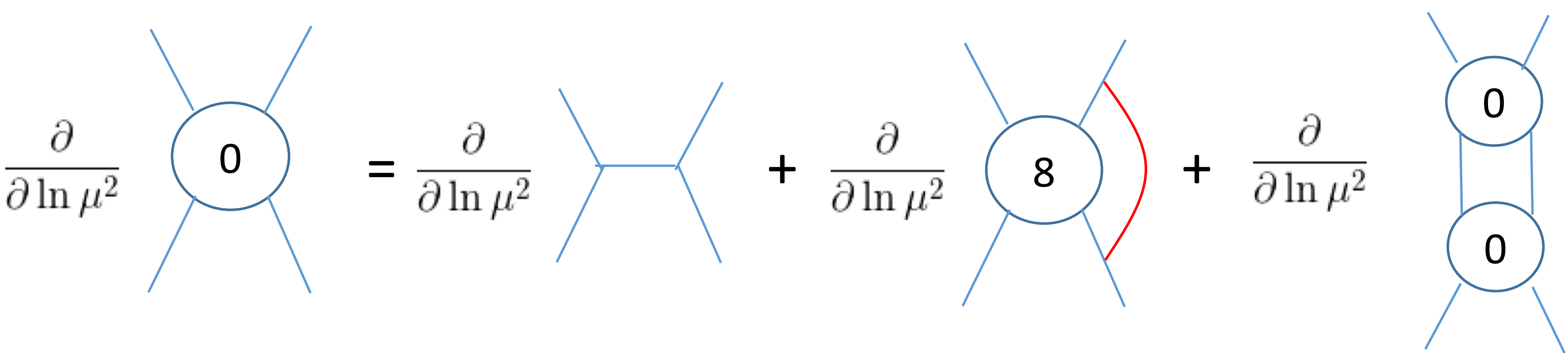
$$F_0 = \frac{g^2}{\omega} M_0 - \frac{g^2}{2\pi^2\omega^2} F_8 G_0 + \frac{1}{8\pi^2\omega} F_0^2$$

$$F_8 = \frac{g^2}{\omega} M_8 + \frac{g^2 C_A}{8\pi^2\omega} \frac{d}{d\omega} F_8 + \frac{1}{8\pi^2\omega} F_8^2$$

$$M_0 = \begin{pmatrix} C_F & -2T_f \\ 2C_F & 4C_A \end{pmatrix}$$

$$M_8 = \begin{pmatrix} -1/2N_c & -T_f \\ C_A & 2C_A \end{pmatrix}$$

$$G_0 = \begin{pmatrix} C_F & 0 \\ 0 & C_A \end{pmatrix}$$



Iterative solution of BER

2-loop anomalous dimension

$$F_0(\omega) = \frac{g^2}{\omega} M_0 + \frac{g^4}{8\pi^2\omega^3}(M_0^2 - 4M_8G_0)$$

3-loop

$$+\frac{g^6}{32\pi^4\omega^5}\Big(M_0^3 + 2C_A M_8 G_0 - 2M_8^2 G_0$$

$$-2M_0M_8G_0 - 2M_8G_0M_0\Big).$$

Agrees with the DLA part of the complete 2x2 anomalous dimension matrix directly computed to 3 loops by Moch, Vermaseren, Vogt (2014)

Final result (4 flavors)

Bartels, Ermolaev, Ryskin (1996)

$$\Delta G(x) \approx -2.29\Delta\Sigma(x) \propto x^{-3.45\sqrt{\frac{\alpha_s N_c}{2\pi}}}$$

KPS approach

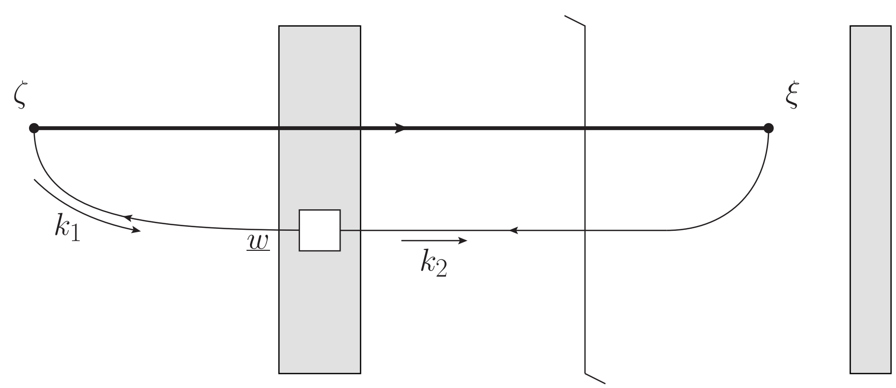
Quark Helicity TMD at Small x

- Analyzing the relevant diagrams we arrive at the quark helicity TMD at small x,

$$g_{1L}^q(x, k_T^2) = \frac{4N_c}{(2\pi)^6} \int d^2\zeta \, d^2w \, d^2y \, e^{-i\underline{k}\cdot(\underline{\zeta}-\underline{y})} \int\limits_{\Lambda^2/s}^{1} \frac{dz}{z} \, \frac{\underline{\zeta}-\underline{w}}{|\underline{\zeta}-\underline{w}|^2} \cdot \frac{\underline{y}-\underline{w}}{|\underline{y}-\underline{w}|^2} \, G_{\underline{w},\underline{\zeta}}(zs)$$

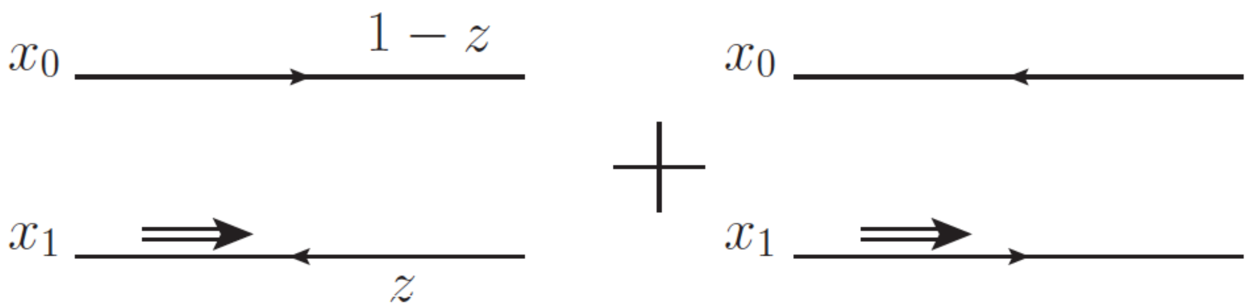
  where $G_{w\zeta}$ is the polarized dipole amplitude (defined on the next slide).

- Here s is the cms energy squared, Λ is some IR cutoff, underlining denotes transverse vectors, z = smallest longitudinal momentum fraction of the dipole momentum out of those carried by the quark and the antiquark



Polarized Dipole

- All flavor-singlet small-x helicity observables depend on one object, "polarized dipole amplitude":



$$G_{10}(z) \equiv \frac{1}{2N_c} \text{Re} \left\langle\!\!\left\langle \text{T}\,\text{tr}\left[V_{\underline{0}} V_{\underline{1}}^{pol\,\dagger}\right] + \text{T}\,\text{tr}\left[V_{\underline{1}}^{pol} V_{\underline{0}}^{\dagger}\right] \right\rangle\!\!\right\rangle(z)$$

unpolarized quark

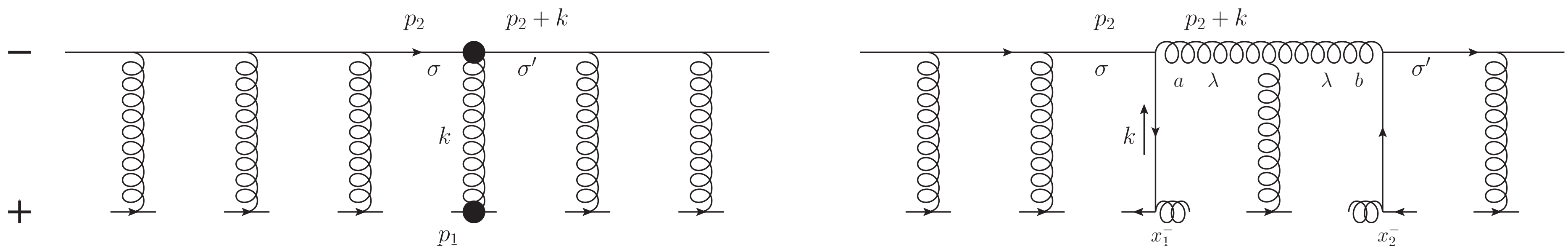
polarized quark: eikonal propagation, non-eikonal spin-dependent interaction

$$V_{\underline{x}} \equiv \mathcal{P} \exp \left[ig \int\limits_{-\infty}^{\infty} dx^+ \, A^-(x^+, 0^-, \underline{x}) \right]$$

- Double brackets denote an object with energy suppression scaled out:

$$\left\langle\!\!\left\langle \mathcal{O} \right\rangle\!\!\right\rangle(z) \equiv zs \left\langle \mathcal{O} \right\rangle(z)$$

Polarized fundamental "Wilson line"



- In the end one arrives at (cf. Chirilli '18)

$$V_{\underline{x}}^{pol} = \frac{igp_1^+}{s} \int\limits_{-\infty}^{\infty} dx^- \, V_{\underline{x}}[+\infty, x^-] \, F^{12}(x^-, \underline{x}) \, V_{\underline{x}}[x^-, -\infty]$$

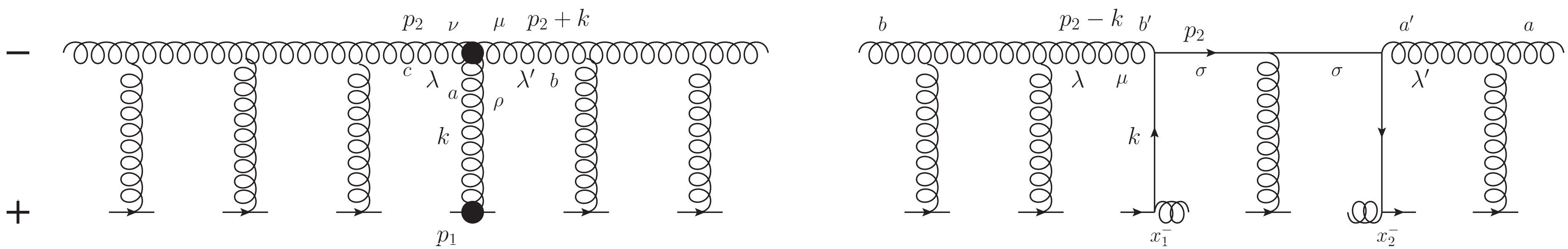
$$- \frac{g^2 \, p_1^+}{s} \int\limits_{-\infty}^{\infty} dx_1^- \int\limits_{x_1^-}^{\infty} dx_2^- \, V_{\underline{x}}[+\infty, x_2^-] \, t^b \, \psi_\beta(x_2^-, \underline{x}) \, U_{\underline{x}}^{ba}[x_2^-, x_1^-] \left[\frac{1}{2} \gamma^+ \gamma^5 \right]_{\alpha\beta} \bar{\psi}_\alpha(x_1^-, \underline{x}) \, t^a \, V_{\underline{x}}[x_1^-, -\infty].$$

- The first term on the right (the gluon exchange contribution) was known before (KPS '17), the second term (quark exchange) is new.
- We have employed an adjoint light-cone Wilson line

$$U_{\underline{x}}[b^-, a^-] = \mathcal{P} \exp \left[ig \int\limits_{a^-}^{b^-} dx^- \, \mathcal{A}^+(x^+ = 0, x^-, \underline{x}) \right]$$

Polarized adjoint “Wilson line”

- Quarks mix with gluons. Therefore, we need to construct the adjoint polarized Wilson line --- the leading helicity-dependent part of the gluon scattering amplitude on the longitudinally polarized target.



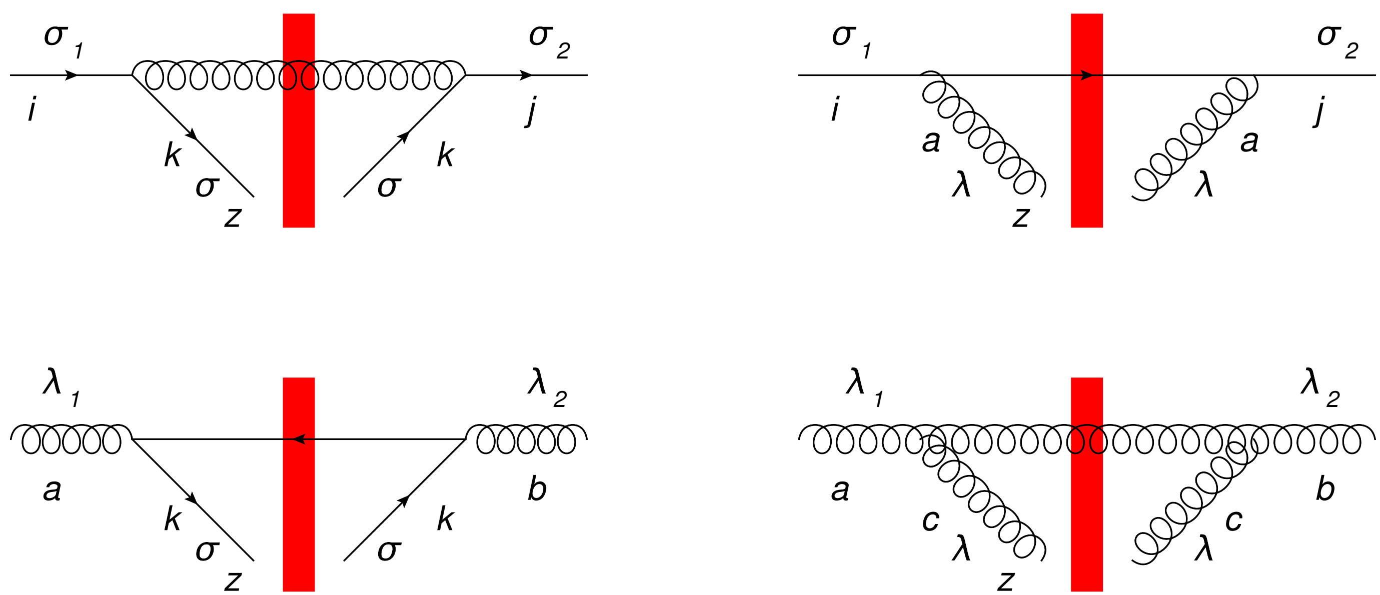
- The calculation is similar to the quark scattering case. It yields (cf. Chirilli ‘18)

$$(U_{\underline{x}}^{pol})^{ab} = \frac{2i\, g\, p_1^+}{s} \int\limits_{-\infty}^{+\infty} dx^- \ \left(U_{\underline{x}}[+\infty, x^-]\, \mathcal{F}^{12}(x^+=0, x^-, \underline{x})\, U_{\underline{x}}[x^-, -\infty] \right)^{ab}$$

$$- \frac{g^2\, p_1^+}{s} \int\limits_{-\infty}^{\infty} dx_1^- \int\limits_{x_1^-}^{\infty} dx_2^-\, U_{\underline{x}}^{aa'}[+\infty, x_2^-]\, \bar{\psi}(x_2^-, \underline{x})\, t^{a'}\, V_{\underline{x}}[x_2^-, x_1^-]\, \frac{1}{2}\gamma^+\gamma_5\, t^{b'}\, \psi(x_1^-, \underline{x})\, U_{\underline{x}}^{b'b}[x_1^-, -\infty] - \text{c.c.}$$

Helicity Evolution Ingredients

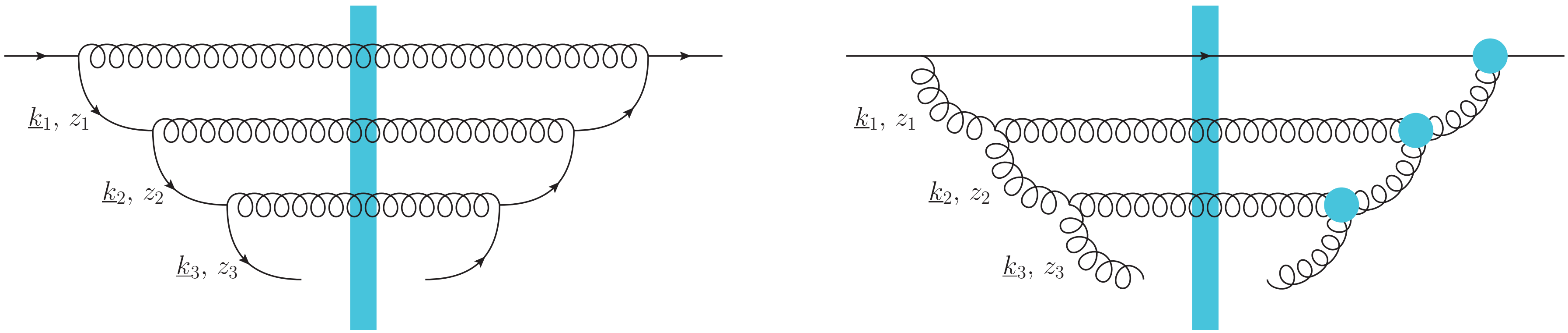
- Unlike the unpolarized evolution, in one step of helicity evolution we may emit a soft gluon or a soft quark (all in $A^+=0$ LC gauge of the projectile):



- When emitting gluons, one gluon is eikonal, while another one is soft, but non-eikonal, as is needed to transfer polarization down the cascade/ladder.

Helicity Evolution: Ladders

- To get an idea of how the helicity evolution works let us try iterating the splitting kernels by considering ladder diagrams (circles denote non-eikonal gluon vertices):

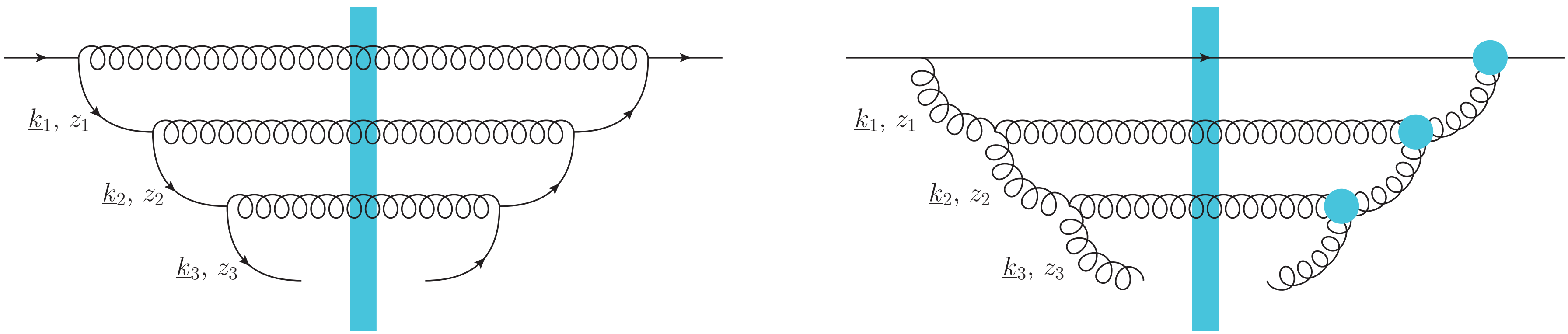


- To get the leading-energy asymptotics we need to order the longitudinal momentum fractions of the quarks and gluons (just like in the unpolarized evolution case) $1 \gg z_1 \gg z_2 \gg z_3 \gg \ldots$

obtaining a nested integral

$$\alpha_s^3 \int\limits_{z_i}^{1} \frac{dz_1}{z_1} \int\limits_{z_i}^{z_1} \frac{dz_2}{z_2} \int\limits_{z_i}^{z_2} \frac{dz_3}{z_3}\, z_3 \otimes \frac{1}{z_3\, s} \sim \frac{1}{s}\, \alpha_s^3\, \ln^3 s$$

Helicity Evolution: Ladders



- However, these are not all the logs of energy one can get here. Transverse momentum (or distance) integrals have UV and IR divergences, which lead to logs of energy as well.
- If we order transverse momenta / distances as (Sudakov-β ordering)

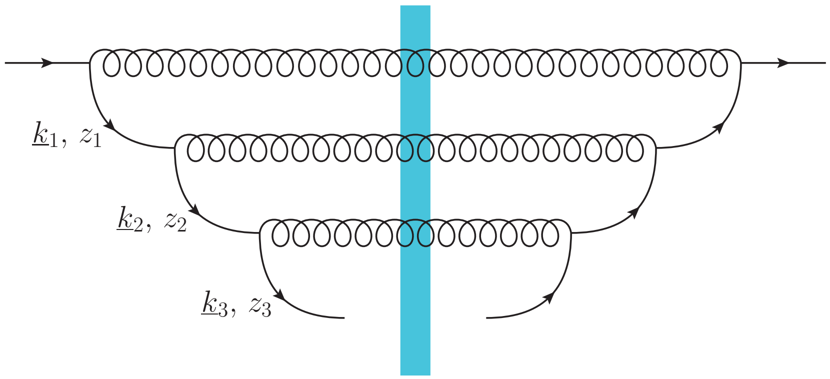
$$\frac{\underline{k}_1^2}{z_1} \ll \frac{\underline{k}_2^2}{z_2} \ll \frac{\underline{k}_3^2}{z_3} \ll \ldots \qquad z_1\, \underline{x}_1^2 \gg z_2\, \underline{x}_2^2 \gg z_3\, \underline{x}_3^2 \gg \ldots$$

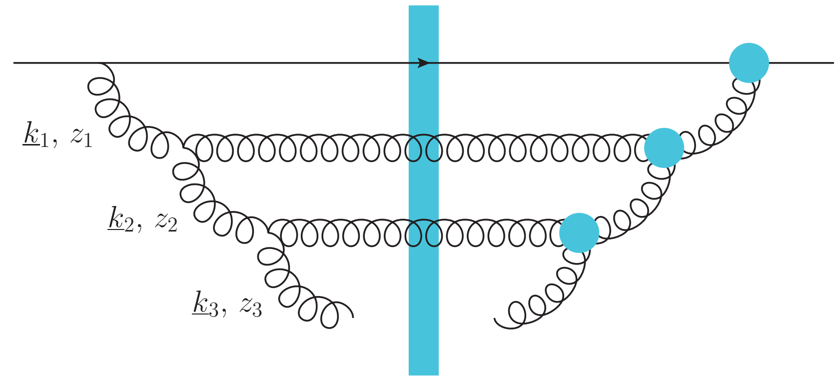
we would get integrals like

$$\int\limits_{1/(z_n\, s)}^{x_{n-1,\perp}^2\, z_{n-1}/z_n} \frac{dx_{n,\perp}^2}{x_{n,\perp}^2}$$

also generating logs of energy.

Helicity Evolution: Ladders





- To summarize, the above ladder diagrams are parametrically of the order

$$\frac{1}{s}\,\alpha_s^3\,\ln^6 s$$

- Note two features:
  - 1/s suppression due to non-eikonal exchange
  - two logs of energy per each power of the coupling!

Resummation Parameter

- For helicity evolution the resummation parameter is different from BK or JIMWLK, which resum powers of leading logarithms (LLA)

$$\alpha_s \ \ln(1/x)$$

- Helicity evolution resummation parameter is double-logarithmic (DLA):

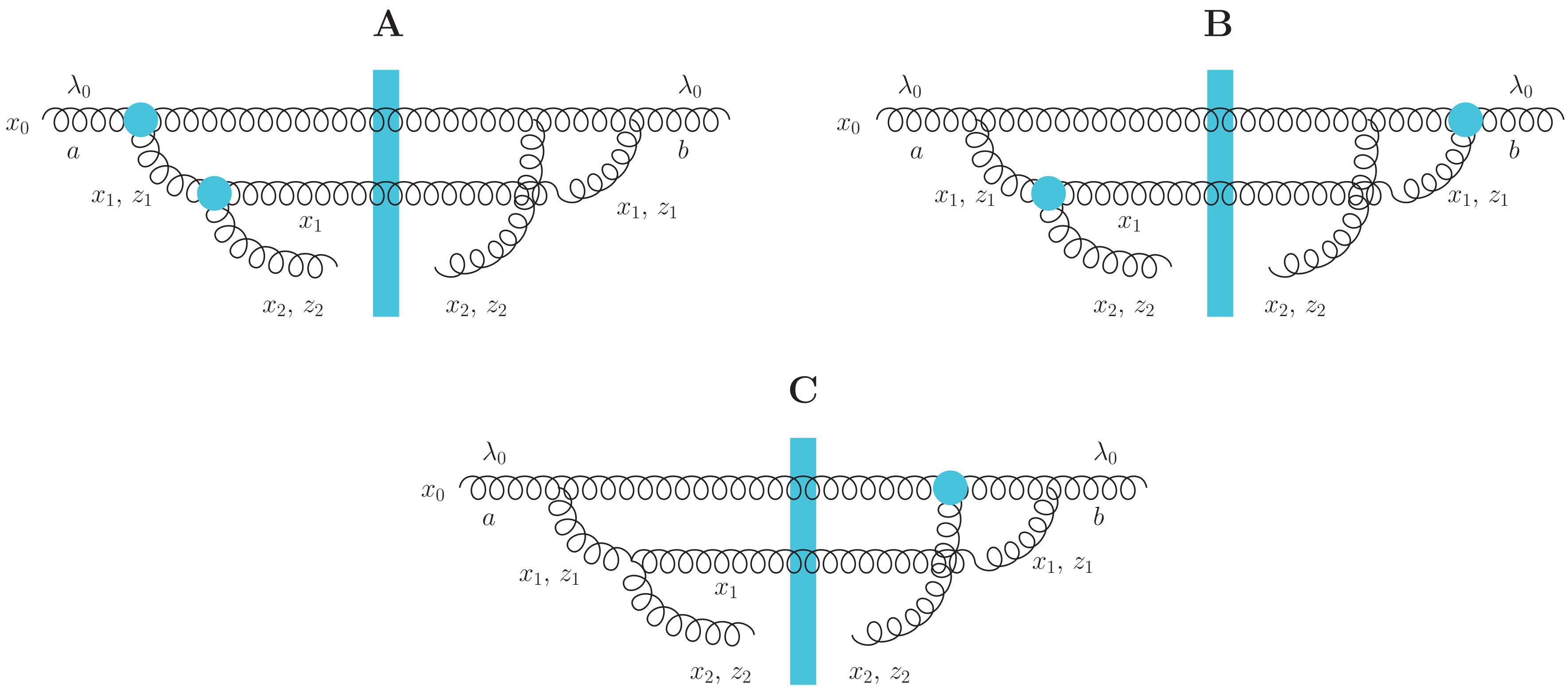
$$\alpha_s \ \ln^2 \frac{1}{x}$$

- The second logarithm of x arises due to transverse momentum integration being logarithmic both in UV and IR.

- This was known before: Kirschner and Lipatov ’83; Kirschner ’84; Bartels, Ermolaev, Ryskin ‘95, ‘96; Griffiths and Ross ’99; Itakura et al ’03; Bartels and Lublinsky ‘03.

Non-Ladder Diagrams

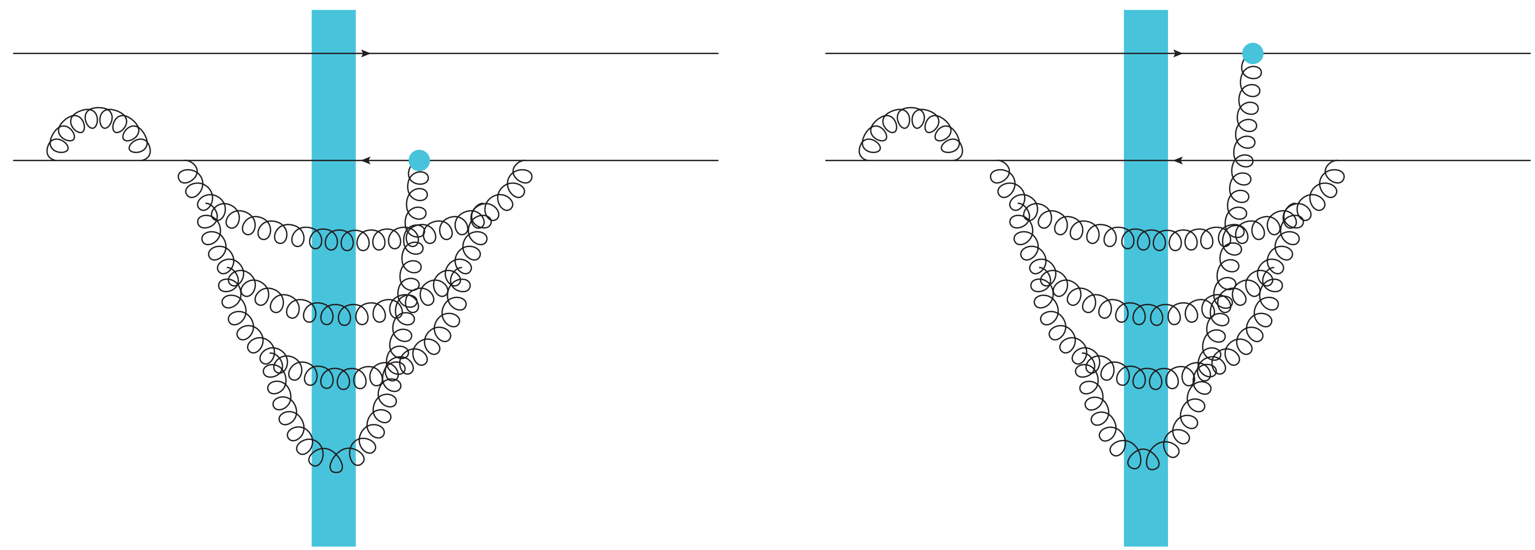
- Ladder diagrams are not the whole story. The non-ladder diagrams below are also leading-order (that is, DLA).



- Non-ladder soft quark emissions cancel for flavor-singlet observables we are primarily interested in. Non-ladder soft gluons do not cancel.

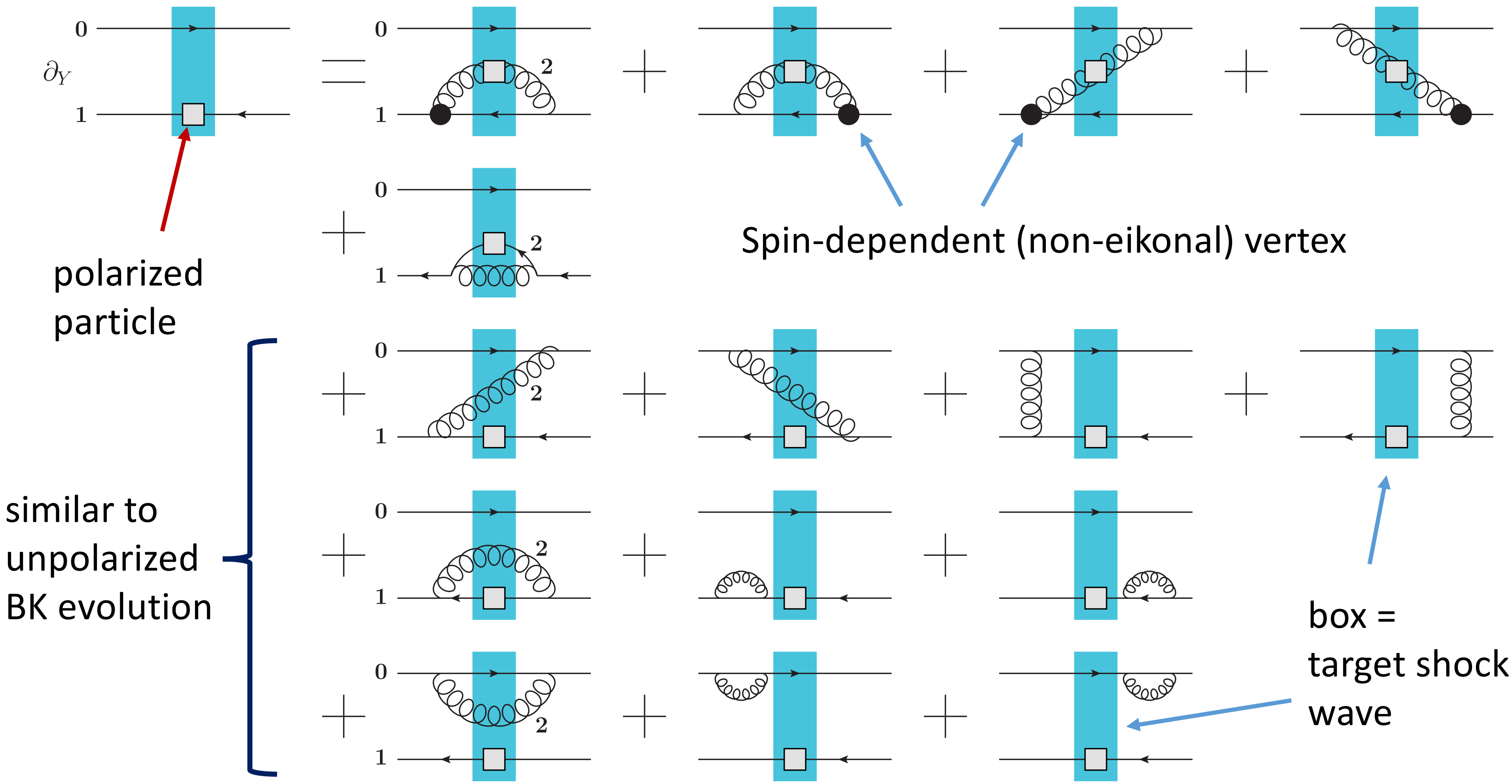
Virtual Corrections

- In addition, virtual corrections from the unpolarized LLA evolution have UV divergences, which cancel between real and virtual diagrams. Here the corrections are not cancelled, but are regulated by the cms energy.
- Helicity evolution thus also contains the following types of graphs:

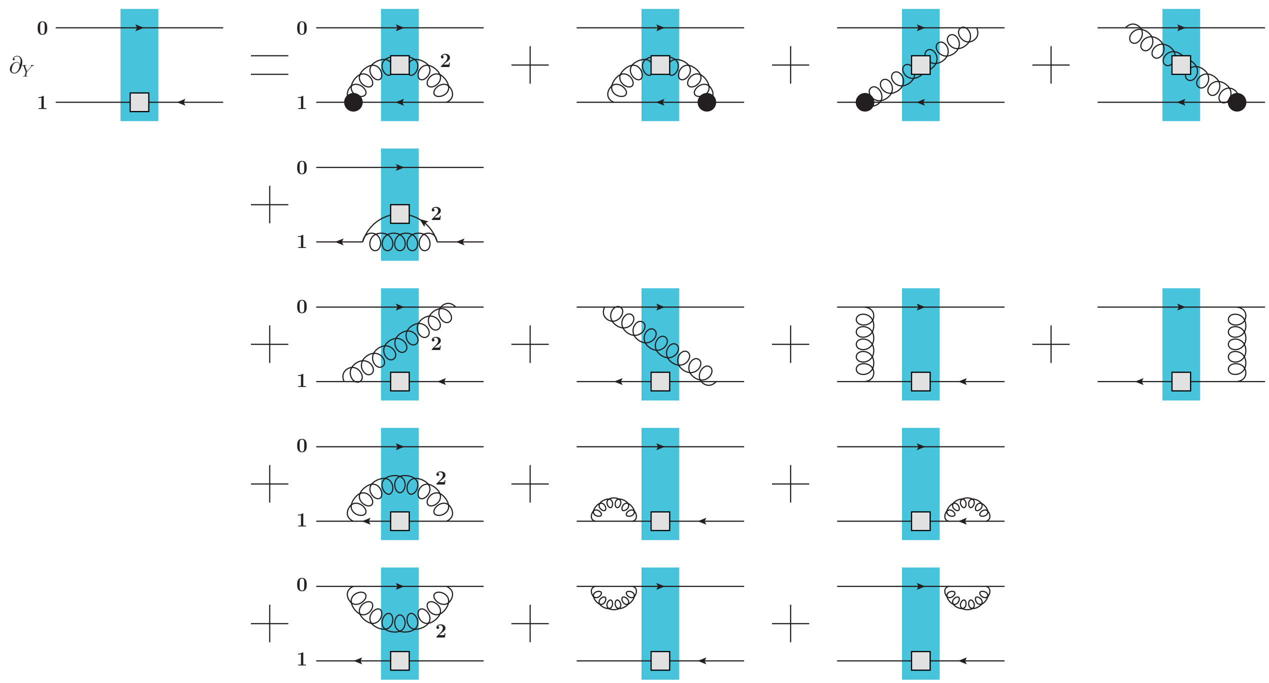


Evolution for Polarized Quark Dipole

One can construct an evolution equation for the polarized dipole:



Evolution for Polarized Quark Dipole



$$\langle\langle \ldots \rangle\rangle = \frac{1}{z\, s} \langle \ldots \rangle$$

$$\rho'^{\,2} = \frac{1}{z'\, s}$$

$$\frac{1}{N_c} \left\langle\!\!\left\langle \mathrm{tr}\left[V_0^{unp}\, V_1^{pol\,\dagger}\right]\right\rangle\!\!\right\rangle(z) = \frac{1}{N_c} \left\langle\!\!\left\langle \mathrm{tr}\left[V_0^{unp}\, V_1^{pol\,\dagger}\right]\right\rangle\!\!\right\rangle_0(z) + \frac{\alpha_s}{2\pi^2} \int\limits_{z_i}^{z} \frac{dz'}{z'} \int\limits_{\rho'^2} \frac{d^2 x_2}{x_{21}^2}$$

$$\times \Bigg\{ \theta(x_{10} - x_{21})\, \frac{2}{N_c} \left\langle\!\!\left\langle \mathrm{tr}\left[t^b\, V_0^{unp}\, t^a\, V_1^{unp\,\dagger}\right]\, U_2^{pol\, ba} \right\rangle\!\!\right\rangle(z')$$

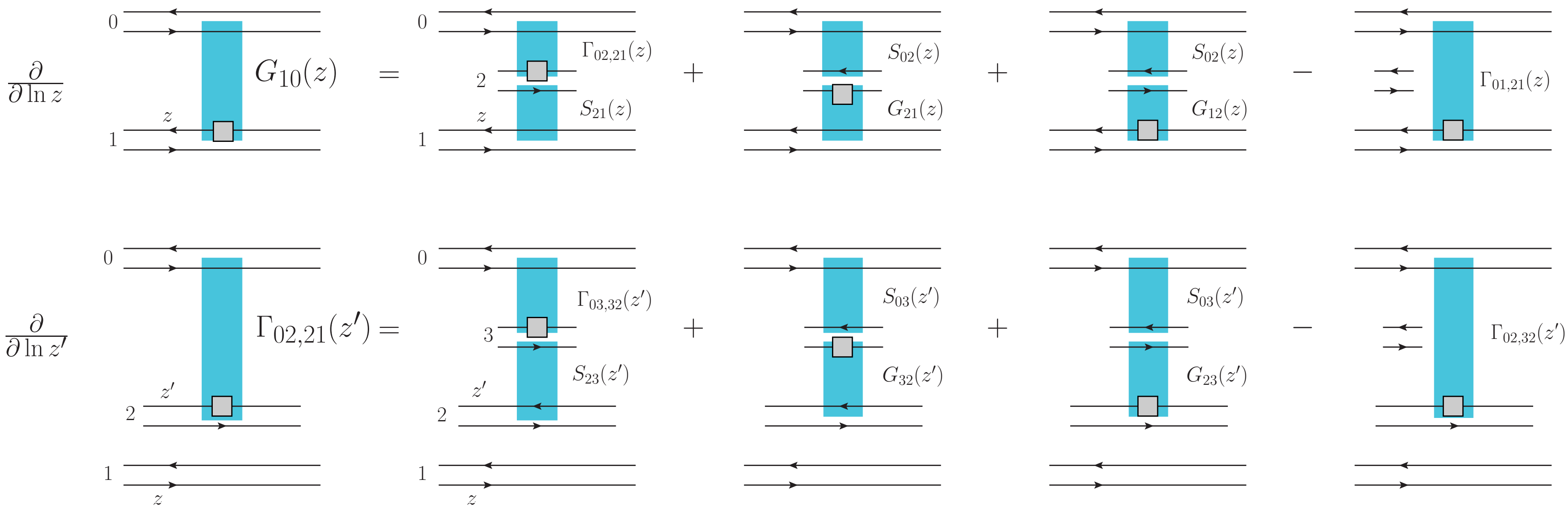
$$+ \theta(x_{10}^2 z - x_{21}^2 z')\, \frac{1}{N_c} \left\langle\!\!\left\langle \mathrm{tr}\left[t^b\, V_0^{unp}\, t^a\, V_2^{pol\,\dagger}\right]\, U_1^{unp\, ba} \right\rangle\!\!\right\rangle(z')$$

$$+ \theta(x_{10} - x_{21})\, \frac{1}{N_c} \left[\left\langle\!\!\left\langle \mathrm{tr}\left[V_0^{unp}\, V_2^{unp\,\dagger}\right]\, \mathrm{tr}\left[V_2^{unp}\, V_1^{pol\,\dagger}\right] \right\rangle\!\!\right\rangle(z') - N_c \left\langle\!\!\left\langle \mathrm{tr}\left[V_0^{unp}\, V_1^{pol\,\dagger}\right] \right\rangle\!\!\right\rangle(z') \right] \Bigg\}$$

Equation does not close!

Polarized Dipole Evolution in the Large-N$_c$ Limit

In the large-N$_c$ limit the equations close, leading to a system of 2 equations:



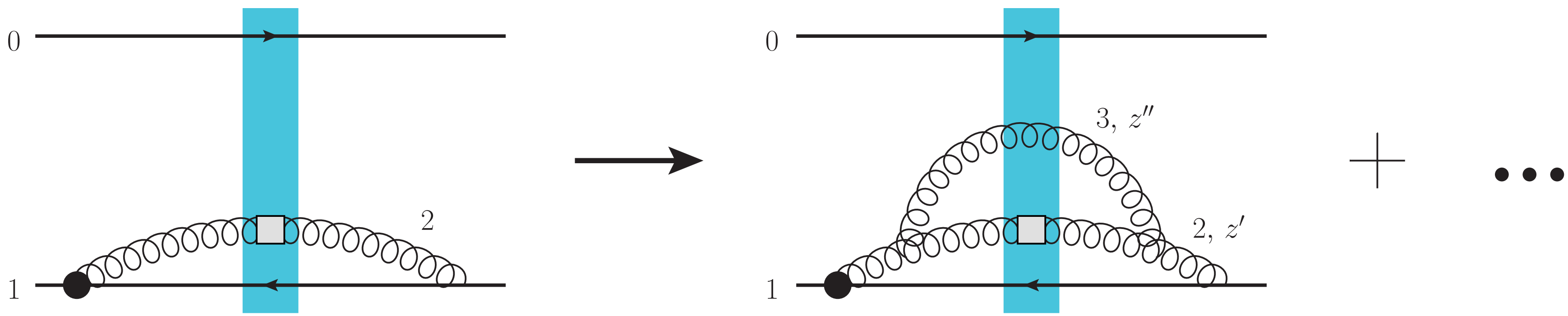
$$G_{10}(z) = G_{10}^{(0)}(z) + \frac{\alpha_s N_c}{2\pi} \int\limits_{z_i}^{z} \frac{dz'}{z'} \int\limits_{\rho'^2}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[2\,\Gamma_{02,\,21}(z')\,S_{21}(z') + 2\,G_{21}(z')\,S_{02}(z') + G_{12}(z')\,S_{02}(z') - \Gamma_{01,\,21}(z')\right]$$

$$\Gamma_{02,\,21}(z') = \Gamma_{02,\,21}^{(0)}(z') + \frac{\alpha_s N_c}{2\pi} \int\limits_{z_i}^{z'} \frac{dz''}{z''} \int\limits_{\rho''^2}^{\min\{x_{02}^2, x_{21}^2\, z'/z''\}} \frac{dx_{32}^2}{x_{32}^2} \left[2\,\Gamma_{03,\,32}(z'')\,S_{23}(z'') + 2\,G_{32}(z'')\,S_{03}(z'') + G_{23}(z'')\,S_{03}(z'') - \Gamma_{02,\,32}(z'')\right]$$

S = found from BK/JIMWLK, it is LLA

"Neighbor" dipole

- There is a new object in the evolution equation – **the neighbor dipole**.
- This is specific for the DLA evolution. Gluon emission may happen in one dipole, but, due to transverse distance ordering, may `know' about another dipole:

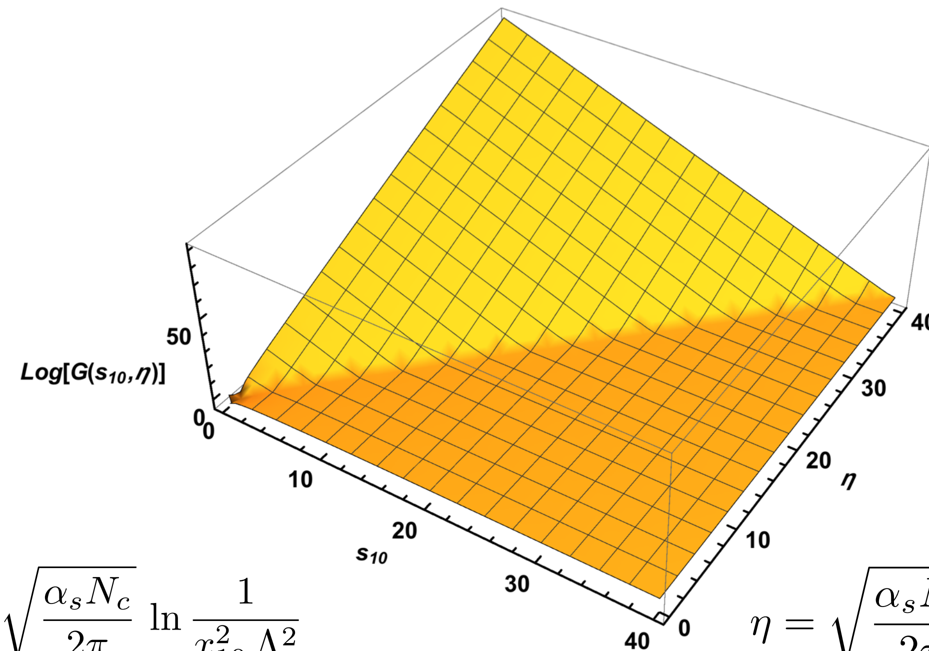


$$x_{21}^2\, z' \gg x_{32}^2\, z''$$

- We denote the evolution in the neighbor dipole 02 by $\Gamma_{02,\,21}(z')$

Quark Helicity at Small x

- These equations can be solved both numerically and analytically. (KPS '16-'17)



$$s_{10} = \sqrt{\frac{\alpha_s N_c}{2\pi}} \ln \frac{1}{x_{10}^2 \Lambda^2} \qquad \eta = \sqrt{\frac{\alpha_s N_c}{2\pi}} \ln \frac{zs}{\Lambda^2}$$

- The small-x asymptotics of quark helicity is (at large N_c)

$$\Delta q(x, Q^2) \sim \left(\frac{1}{x}\right)^{\alpha_h^q} \quad \text{with} \quad \alpha_h^q = \frac{4}{\sqrt{3}} \sqrt{\frac{\alpha_s N_c}{2\pi}} \approx 2.31 \sqrt{\frac{\alpha_s N_c}{2\pi}}$$

Conclusions

- We conclude that the small-x asymptotics of gluon helicity (at large N_c) is

$$\Delta G(x, Q^2) \sim \left(\frac{1}{x}\right)^{\alpha_h^G} \quad \text{with} \quad \alpha_h^G = \frac{13}{4\sqrt{3}} \sqrt{\frac{\alpha_s N_c}{2\pi}} \approx 1.88 \sqrt{\frac{\alpha_s N_c}{2\pi}}$$

while the quark helicity asymptotics is

$$\Delta q(x, Q^2) \sim \left(\frac{1}{x}\right)^{\alpha_h^q} \quad \text{with} \quad \alpha_h^q = \frac{4}{\sqrt{3}} \sqrt{\frac{\alpha_s N_c}{2\pi}} \approx 2.31 \sqrt{\frac{\alpha_s N_c}{2\pi}}$$

- ΔG can be further measured from A_{LL} in pp and at EIC. One may use our approach to combine experiment and theory to constrain the quark and gluon spin (and OAM) at small x (in progress, a long-term goal).

- Preliminary results indicate a possible enhancement of quark and gluon spin as compared to DSSV.

OAM at small-x

`PDF' of OAM

Define the x-distribution $L_{can} = \int dx L_{can}(x)$.

Hagler, Schafer (1998)
Harindranath, Kundu (1999)
YH, Yoshida (2012)

$$L^q_{can}(x) = \int d^2b_\perp d^2k_\perp (\vec{b}_\perp \times \vec{k}_\perp)_z W^q(x, \vec{b}_\perp, \vec{k}_\perp)$$

Wigner distribution

Not a usual (twist-2) PDF.
It's a twist-3 PDF, similar to $g_2(x)$.

Twist structure of OAM distribution

YH, Yoshida (2012)

Wandzura-Wilczek part

$$L^q_{can}(x) = x\int_x^{\epsilon(x)} \frac{dx'}{x'}(H_q(x')+E_q(x')) - x\int_x^{\epsilon(x)} \frac{dx'}{x'^2}\tilde{H}_q(x')$$

$$-x\int_x^{\epsilon(x)} dx_1 \int_{-1}^{1} dx_2 \Phi_F(x_1,x_2)\mathcal{P}\frac{3x_1-x_2}{x_1^2(x_1-x_2)^2}$$

$$-x\int_x^{\epsilon(x)} dx_1 \int_{-1}^{1} dx_2 \tilde{\Phi}_F(x_1,x_2)\mathcal{P}\frac{1}{x_1^2(x_1-x_2)}\,.$$

genuine twist-3

$$\Phi_F \sim \langle P'|\bar{\psi}\gamma^+ F^{+i}\psi|P\rangle$$

$$M_F \sim \langle P'|F^{+\mu}F^{+i}F^+_\mu|P\rangle$$

$$L^g_{can}(x) = \frac{x}{2}\int_x^{\epsilon(x)} \frac{dx'}{x'^2}(H_g(x')+E_g(x')) - x\int_x^{\epsilon(x)} \frac{dx'}{x'^2}\Delta G(x')$$

$$+2x\int_x^{\epsilon(x)} \frac{dx'}{x'^3}\int dX\,\Phi_F(X,x') + 2x\int_x^{\epsilon(x)} dx_1 \int_{-1}^{1} dx_2 \tilde{M}_F(x_1,x_2)\mathcal{P}\frac{1}{x_1^3(x_1-x_2)}$$

$$+2x\int_x^{\epsilon(x)} dx_1 \int_{-1}^{1} dx_2 M_F(x_1,x_2)\mathcal{P}\frac{2x_1-x_2}{x_1^3(x_1-x_2)^2}$$

QCD evolution of OAM distributions $L_{q,g}(x)$

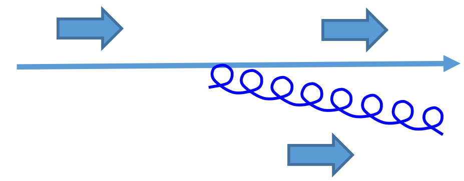
Hagler, Schafer (1998)
Harindranath, Kundu (1999)
Hoodbhoy, Ji, Lu (1999)

$$\frac{d}{d\ln Q^2}\begin{pmatrix} L_q(x) \\ L_g(x) \end{pmatrix} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{dz}{z}\begin{pmatrix} \hat{P}_{qq}(z) & \hat{P}_{qg}(z) & \Delta\hat{P}_{qq}(z) & \Delta\hat{P}_{qg}(z) \\ \hat{P}_{gq}(z) & \hat{P}_{gg}(z) & \Delta\hat{P}_{gq}(z) & \Delta\hat{P}_{gg}(z) \end{pmatrix}\begin{pmatrix} L_q(x/z) \\ L_g(x/z) \\ \Delta q(x/z) \\ \Delta G(x/z) \end{pmatrix},$$

$$\begin{aligned}
\hat{P}_{qq}(z) &= C_F\left(\frac{z(1+z^2)}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right), \\
\hat{P}_{qg}(z) &= n_f z(z^2 + (1-z)^2), \\
\hat{P}_{gq}(z) &= C_F(1 + (1-z)^2), \\
\hat{P}_{gg}(z) &= 6\frac{(z^2-z+1)^2}{(1-z)_+} + \frac{\beta_0}{2}\delta(z-1), \\
\Delta\hat{P}_{qq}(z) &= C_F(z^2-1), \\
\Delta\hat{P}_{qg}(z) &= n_f(1-3z+4z^2-2z^3), \\
\Delta\hat{P}_{gq}(z) &= C_F(-z^2+3z-2), \\
\Delta\hat{P}_{gg}(z) &= 6(z-1)(z^2-z+2),
\end{aligned}$$

Valid only for the Wandzura-Wilczek part

OAM at small-x



$x \ll 1$

Suppose a quark emits a very soft gluon.
Nothing happens to the quark.

From angular momentum conservation, spin and OAM of the emitted gluon have to cancel.

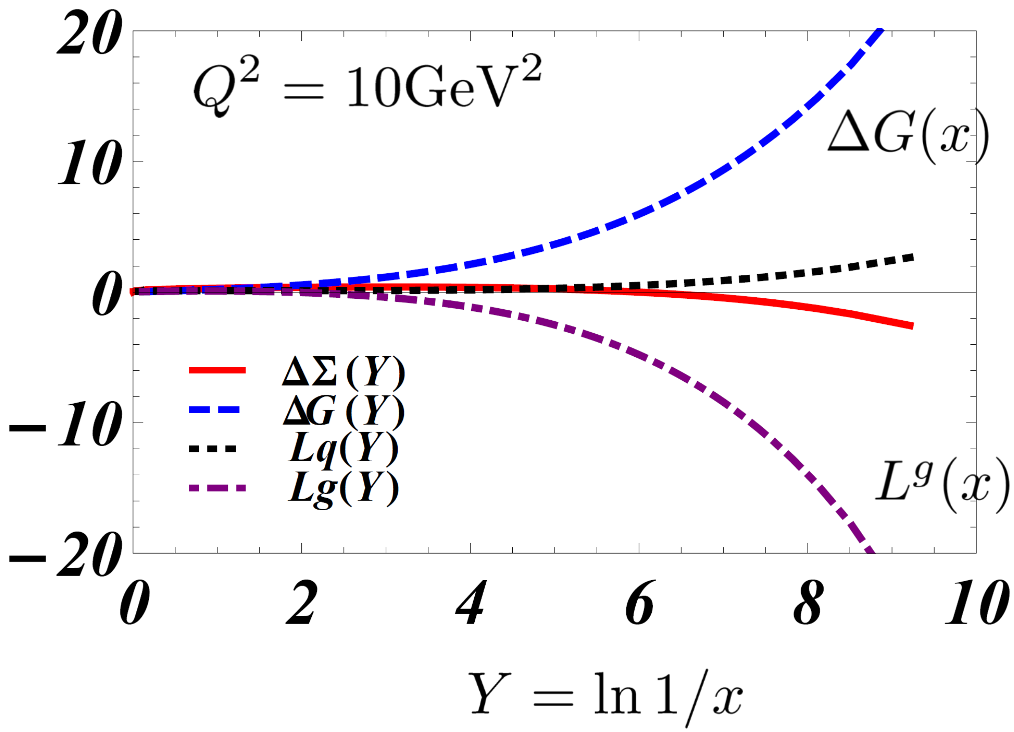
$$\frac{d}{d\ln Q^2} L_g(x) = \frac{\alpha_s}{2\pi}\int_x^1 \frac{dz}{z}(-2C_F + \cdots)\Delta q(x/z)$$

$$\frac{d}{d\ln Q^2} \Delta G(x) = \frac{\alpha_s}{2\pi}\int_x^1 \frac{dz}{z}(+2C_F + \cdots)\Delta q(x/z)$$

1-loop DGLAP evolution

YH, Yang (2018)

Almost complete cancellation of angular momentum between helicity and OAM



All-order argument

Start from the exact formula   Hatta, Yoshida (2013)

$$L^g_{can}(x) = \frac{x}{2}\int_x^{\epsilon(x)} \frac{dx'}{x'^2}(H_g(x') + E_g(x')) - x\int_x^{\epsilon(x)} \frac{dx'}{x'^2}\Delta G(x')$$
$$+2x\int_x^{\epsilon(x)} \frac{dx'}{x'^3}\int dX\Phi_F(X,x') + 2x\int_x^{\epsilon(x)} dx_1\int_{-1}^{1} dx_2 \tilde{M}_F(x_1,x_2)\mathcal{P}\frac{1}{x_1^3(x_1-x_2)}$$
$$+2x\int_x^{\epsilon(x)} dx_1\int_{-1}^{1} dx_2 M_F(x_1,x_2)\mathcal{P}\frac{2x_1-x_2}{x_1^3(x_1-x_2)^2}$$

Assume that the helicity term dominates on the rhs
(in the spirit of double-log approximation)

If $\Delta G(x) \sim \frac{1}{x^\alpha}$, then $L_g(x) \approx -\frac{2}{1+\alpha}\Delta G(x)$

Generalizing IREE to the OAM sector

Boussarie, YH, Yuan (2019)

$$F_0 = \frac{g^2}{\omega} M_0 - \frac{g^2}{2\pi^2\omega^2} F_8 G_0 + \frac{1}{8\pi^2\omega} F_0^2$$

$$F_8 = \frac{g^2}{\omega} M_8 + \frac{g^2 C_A}{8\pi^2\omega} \frac{d}{d\omega} F_8 + \frac{1}{8\pi^2\omega} F_8^2$$

The same coupled equations, but now with 4x4 matrices.

$$M_0 = \begin{pmatrix} C_F & -2T_f & 0 & 0 \\ 2C_F & 4C_A & 0 & 0 \\ -C_F & 2T_f & 0 & 0 \\ -2C_F & -4C_A & 2C_F & 2C_A \end{pmatrix} \quad G_0 = \begin{pmatrix} C_F & 0 & 0 & 0 \\ 0 & C_A & 0 & 0 \\ 0 & 0 & C_F & 0 \\ 0 & 0 & 0 & C_A \end{pmatrix} \quad M_8 = \begin{pmatrix} -1/2N_c & -T_f & 0 & 0 \\ C_A & 2C_A & 0 & 0 \\ 1/2N_c & T_f & 0 & 0 \\ -C_A & -2C_A & C_A & C_A \end{pmatrix}$$

Exact solution

$$F_0^{2\times 2} = \frac{g^2}{\omega} M_0^{2\times 2} + \begin{pmatrix} A_1 & A_2 \\ B_1 & B_2 \end{pmatrix}$$

← Bartels, Ermolaev, Ryskin solution

$$F_0^{4\times 4} = \frac{g^2}{\omega} M_0^{4\times 4} + \begin{pmatrix} A_1 & A_2 & 0 & 0 \\ B_1 & B_2 & 0 & 0 \\ -A_1 & -A_2 & 0 & 0 \\ -2B_1 & -2B_2 & 0 & 0 \end{pmatrix}$$

$\times(-1)$

$\times(-2)$

Final result $n_f = 4$

$$\Delta G(x) \approx -2.29 \Delta\Sigma(x) \propto x^{-3.45\sqrt{\frac{\alpha_S N_C}{2\pi}}} \sim \frac{1}{x^{1.01}},$$

$$L_g(x) \approx -2\Delta G(x), \quad \Delta\Sigma(x) \approx -L_q(x).$$

Quark OAM: Definition

- We begin by writing the (Jaffe-Manohar) quark OAM in terms of the Wigner distribution as

$$L_z = \int \frac{d^2 b_\perp db^- \, d^2 k_\perp \, dk^+}{(2\pi)^3} \, (\underline{b} \times \underline{k})_z \, W(k,b)$$

with the quark SIDIS Wigner distribution

$$W^{q,SIDIS}(k,b) = 2 \sum_X \int d^2 r \, dr^- \; e^{ik\cdot r} \left\langle \bar{\psi}_\alpha \left(b - \tfrac{1}{2} r\right) \, V_{\underline{b} - \frac{1}{2}\underline{r}} \left[b^- - \tfrac{1}{2} r^-, \infty\right] |X\rangle \left(\tfrac{1}{2}\gamma^+\right)_{\alpha\beta} \right.$$
$$\left. \times \; \langle X | \, V_{\underline{b} + \frac{1}{2}\underline{r}} \left[\infty, b^- + \tfrac{1}{2} r^-\right] \; \psi_\beta \left(b + \tfrac{1}{2} r\right) \right\rangle$$

- Here, and above, the angle brackets denote ”CGC averaging” in the (polarized) proton target:

$$\left\langle \hat{\mathcal{O}}(b,r) \right\rangle = \frac{1}{2P^+} \int \frac{d^2 \Delta \, d\Delta^+}{(2\pi)^3} \, e^{ib\cdot\Delta} \left\langle P + \frac{\Delta}{2} \middle| \hat{\mathcal{O}}(0,r) \middle| P - \frac{\Delta}{2} \right\rangle$$

Quark OAM: small-x expression

- After some algebra we arrive at the following small-x expression for quark OAM:

$$L_{q+\bar{q}}(x, Q^2) = \frac{8N_c}{(2\pi)^5} \int d^2k_\perp \, d^2x_{10} \, d^2x_1 \, e^{i\underline{k}\cdot\underline{x}_{10}} \, \frac{\underline{x}_{10}}{x_{10}^2} \times \frac{\underline{k}}{\underline{k}^2} \, \underline{x}_1 \times \underline{k} \int\limits_{\Lambda^2/s}^{1} \frac{dz}{z} \, G_{10}(zs) - \sum_f [\Delta q^f(x, Q^2) + \Delta \bar{q}^f(x, Q^2)]$$

- The result is written in terms of the polarized dipole amplitude $G_{10}(z)$. It seems we are done, right?
- This is almost correct. The remaining minor technicality is that the above quark OAM depends on the "first moment" of the polarized dipole amplitude

$$I^k(\underline{x}_{10}, zs) = \int d^2x_1 \, x_1^k \, G_{10}(zs)$$

while all our earlier results for the quark helicity were derived for the "zeroth moment", the impact-parameter integrated polarized dipole amplitude

$$G(x_{10}^2, zs) = \int d^2x_1 \, G_{10}(zs)$$

Quark OAM: small-x asymptotics

- It turns out that the “first moment” of the polarized amplitude is subleading. It grows with energy as a smaller power of energy

$$I^k(\underline{x}_{10}, zs) \sim \left(zs x_{10}^2\right)^{2\sqrt{\frac{\alpha_s N_c}{2\pi}}}$$

than the flavor-singlet quark helicity distribution

$$\Delta\Sigma(x, Q^2) = \sum_f [\Delta q^f(x, Q^2) + \Delta \bar{q}^f(x, Q^2)] \sim \left(\frac{1}{x}\right)^{\alpha_h^q} = \left(\frac{1}{x}\right)^{\frac{4}{\sqrt{3}}\sqrt{\frac{\alpha_s N_c}{2\pi}}} \approx \left(\frac{1}{x}\right)^{2.31\sqrt{\frac{\alpha_s N_c}{2\pi}}}$$

- Since 2.31 > 2, we get (cf. Y. Hatta & D.-J. Yang, 2018)

$$L_{q+\bar{q}}(x, Q^2) = -\Delta\Sigma(x, Q^2) \sim \left(\frac{1}{x}\right)^{\frac{4}{\sqrt{3}}\sqrt{\frac{\alpha_s N_c}{2\pi}}}$$

- Note that this is not a complete cancellation, the contribution to the proton spin is

$$\frac{1}{2}\,\Delta\Sigma(x, Q^2) + L_{q+\bar{q}}(x, Q^2) = -\frac{1}{2}\,\Delta\Sigma(x, Q^2)$$

Gluon OAM: definition

- The gluon OAM story is similar. We start with the Wigner distribution definition

$$L_z = \int \frac{d^2 b_\perp db^- \, d^2 k_\perp \, dk^+}{(2\pi)^3} \, (\underline{b} \times \underline{k})_z \, W(k, b)$$

with the dipole Wigner distribution for gluons

$$W^{G\,dip}(k, b) = \frac{4}{xP^+} \int d\xi^- \, d^2\xi_\perp \;\; e^{ixP^+ \xi^- - i\underline{k}\cdot\underline{\xi}} \times \left\langle \text{tr} \left[F^{+i}(b - \tfrac{1}{2}\xi) \, \mathcal{U}^{[+]}[b - \tfrac{1}{2}\xi, b + \tfrac{1}{2}\xi] \; F^{+i}(b + \tfrac{1}{2}\xi) \, \mathcal{U}^{[-]}[b + \tfrac{1}{2}\xi, b - \tfrac{1}{2}\xi] \right] \right\rangle$$

- We obtain the following expression for the gluon OAM "PDF" (cf. Hatta et al, 2016)

$$L_G(x, Q^2) = \frac{4}{(2\pi)^3 \, x} \int d^2 b_\perp db^- \, d^2 k_\perp \, d\xi^- \, d^2\xi_\perp \;\; (\underline{b} \times \underline{k}) \;\; e^{ixP^+ \xi^- - i\underline{k}\cdot\underline{\xi}} \times \left\langle \text{tr} \left[F^{+i}(b - \tfrac{1}{2}\xi) \, \mathcal{U}^{[+]}[b - \tfrac{1}{2}\xi, b + \tfrac{1}{2}\xi] \; F^{+i}(b + \tfrac{1}{2}\xi) \, \mathcal{U}^{[-]}[b + \tfrac{1}{2}\xi, b - \tfrac{1}{2}\xi] \right] \right\rangle$$

Gluon OAM: small-x asymptotics

- We arrive at the following relation

$$L_G(x, Q^2) = \left(\frac{\alpha_h^q}{4} \, \ln \frac{Q^2}{\Lambda^2} \right) \Delta G(x, Q^2)$$

where

$$\alpha_h^q = \frac{4}{\sqrt{3}} \sqrt{\frac{\alpha_s \, N_c}{2\pi}}$$

- We conclude that

$$L_G(x, Q^2) \sim \Delta G(x, Q^2) \sim \left(\frac{1}{x} \right)^{\alpha_h^G} \sim \left(\frac{1}{x} \right)^{\frac{13}{4\sqrt{3}} \sqrt{\frac{\alpha_s \, N_c}{2\pi}}} \sim \left(\frac{1}{x} \right)^{1.88 \sqrt{\frac{\alpha_s \, N_c}{2\pi}}}$$

- Note that with the DLA accuracy we could also simply conclude that

$$|L_G| \ll |\Delta G|$$

Conclusions

- We have constructed the small-x asymptotics of the quark and gluon OAM (in the Jaffe-Manohar decomposition).

- In the large-N_C limit we obtain

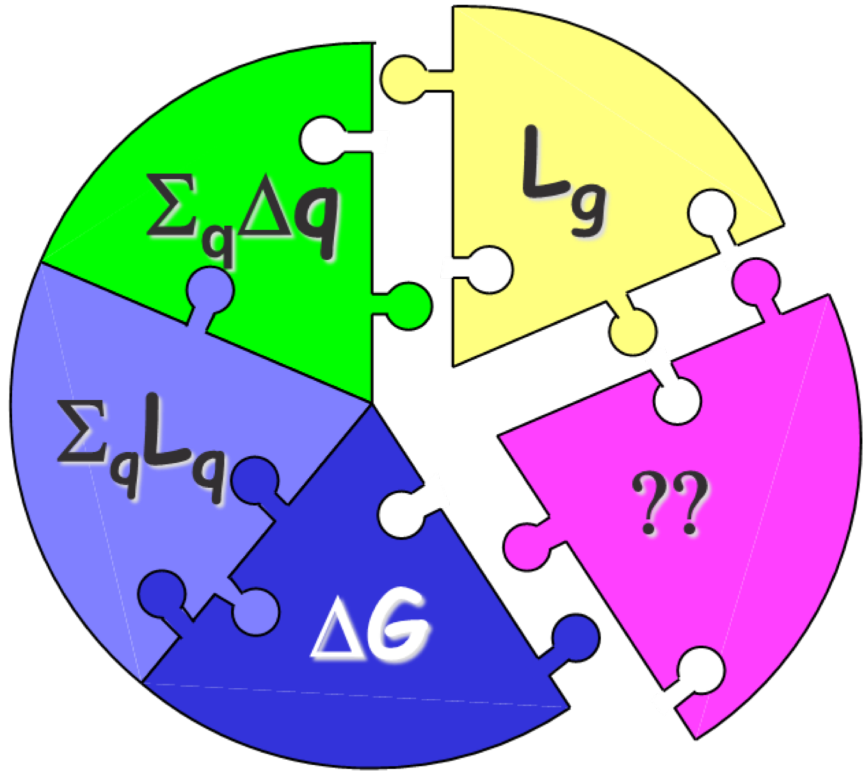
$$L_{q+\bar{q}}(x, Q^2) = -\Delta\Sigma(x, Q^2) \sim \left(\frac{1}{x}\right)^{\frac{4}{\sqrt{3}}\sqrt{\frac{\alpha_s N_c}{2\pi}}},$$

$$L_G(x, Q^2) \sim \Delta G(x, Q^2) \sim \left(\frac{1}{x}\right)^{\frac{13}{4\sqrt{3}}\sqrt{\frac{\alpha_s N_c}{2\pi}}}$$

Backup Slides

Proton Spin Puzzle

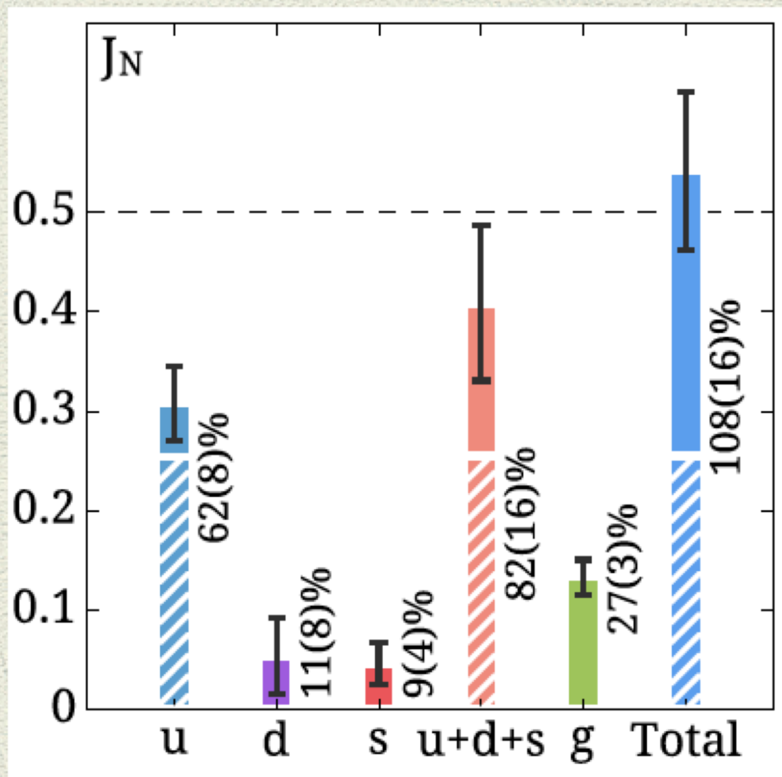
- Current experimental status of helicity PDFs
- OAM
- Lattice results
- Small-x
- What is the roadmap to solving the spin puzzle



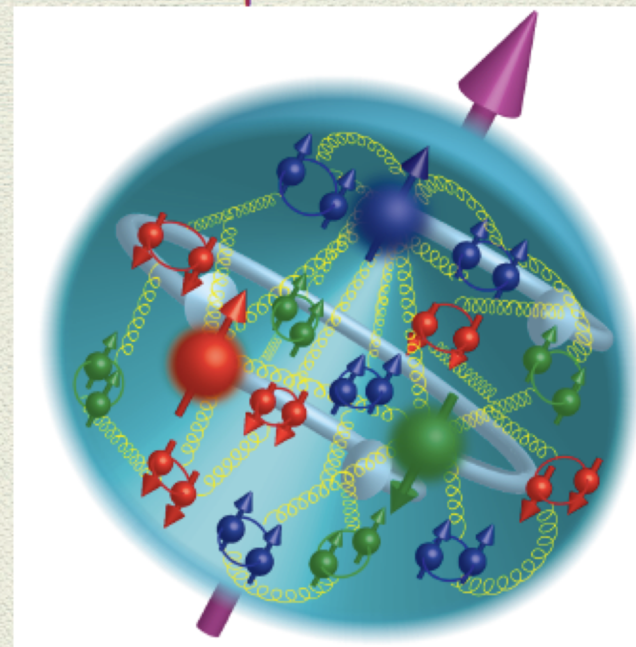
$$S = \frac{1}{2} = \frac{1}{2}\Delta\Sigma + \Delta G + L$$

The proton spin from LQCD

[C. Alexandrou et al., Phys. Rev. Lett. 119, 142002 (2017), [arXiv:1706.02973]]



Better understanding of the spin distribution



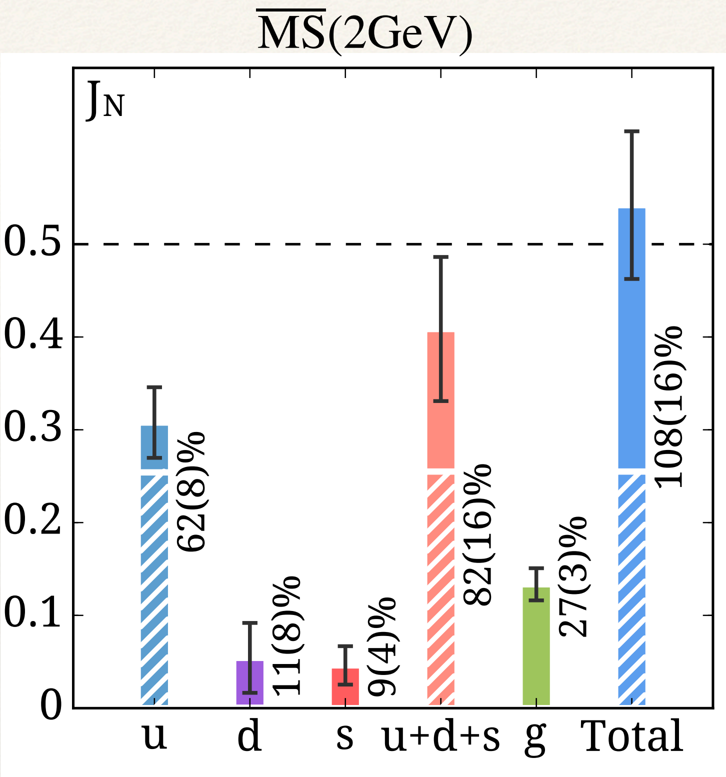
Designed by Z.-E. Meziani

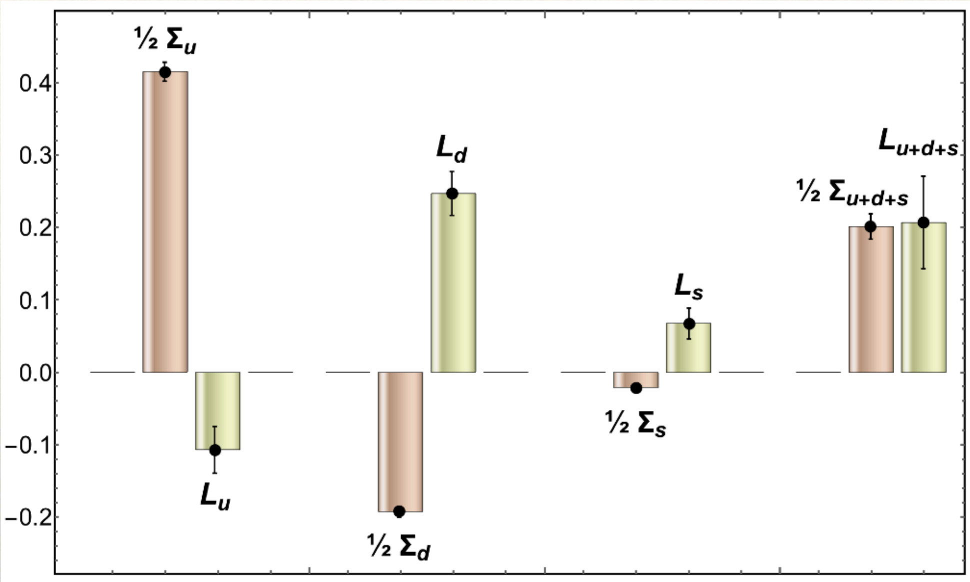
Striped segments: valence quark contributions (connected)
Solid segments: sea quark & gluon contributions (disconnected)

* Satisfaction of spin and momentum sum rule is not forced

Spin decomposition

[C. Alexandrou et al., Phys. Rev. Lett. 119, 142002 (2017), [arXiv:1706.02973]]





Quark orbital angular momentum extracted indirectly ($L_q = J_q - \Sigma_q$)

- ★ Striped segments: connected contributions (connected)
- ★ Solid segments: disconnected contributions (quark & gluon)

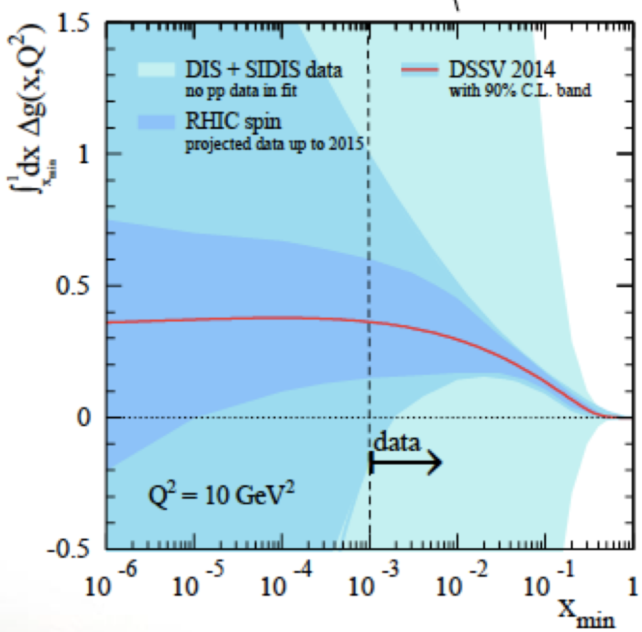
Satisfaction of spin and momentum sum rule is not forced

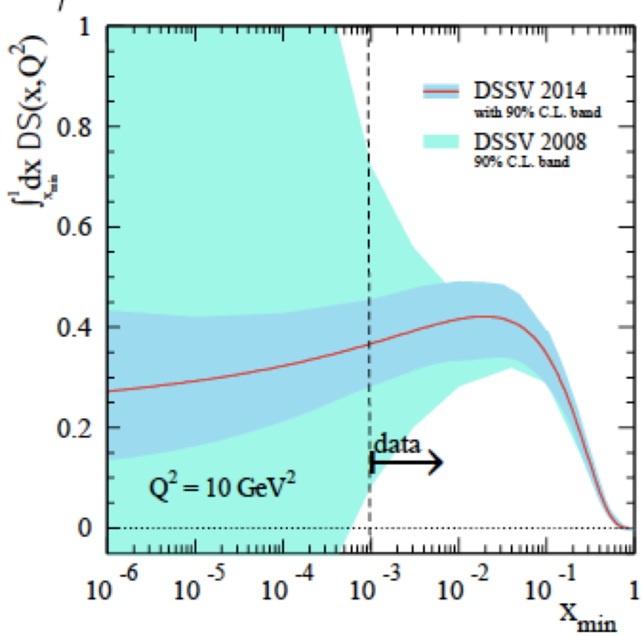
What forms the Spin of the Proton

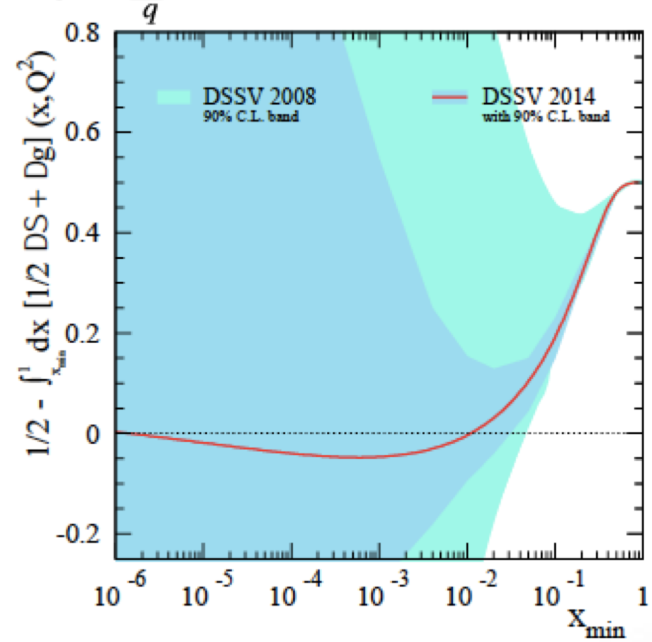
Spin is more than the number $\frac{1}{2}$! It is the interplay between the intrinsic properties and interactions of quarks and gluons

What do we know:

$$\frac{1}{2}\hbar = \left\langle P,\frac{1}{2}\,|\,J^z_{QCD}\,|\,P,\frac{1}{2}\right\rangle = \overbrace{\frac{1}{2}\int_0^1 dx\Delta\Sigma(x,Q^2)}^{\text{total quark spin}} + \overbrace{\int_0^1 dx\Delta G(x,Q^2)}^{\text{gluon spin}} + \overbrace{\int_0^1 dx\left(\sum_q L^z_q + L^z_g\right)}^{\text{angular momentum}}$$







1/2 - Gluon -- Quarks = orbital angular momentum

Gluon: **40%**

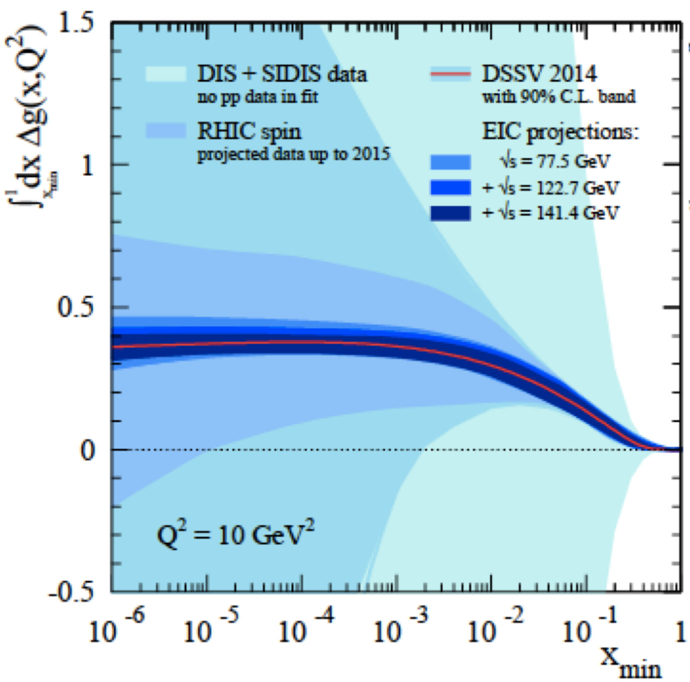
Quarks: **30%**

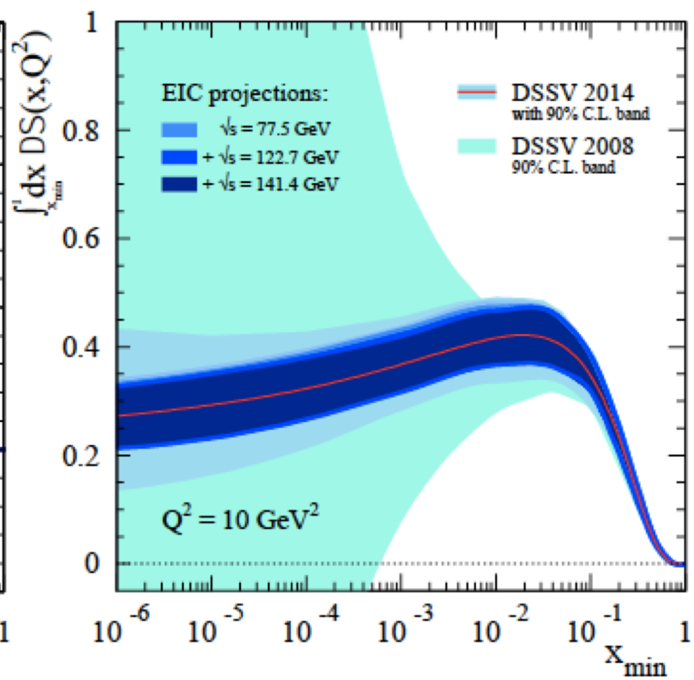
INT EIC program

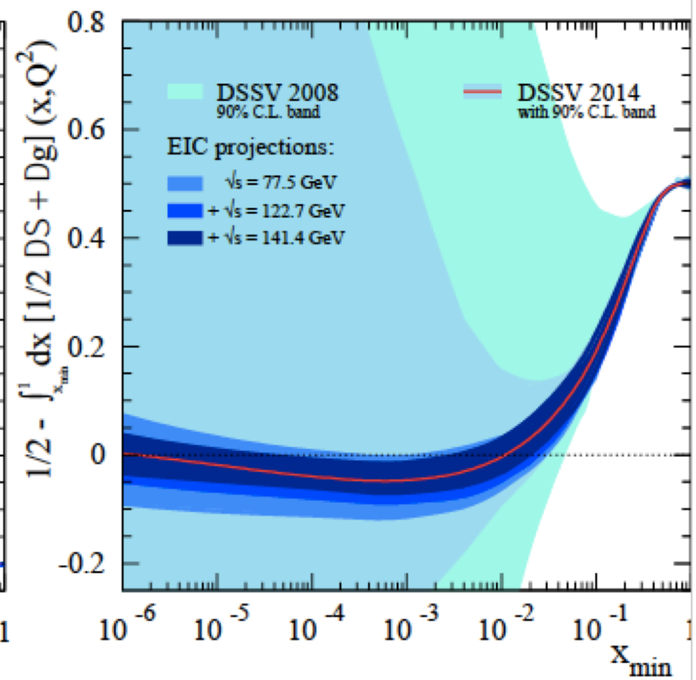
Where does the Spin of the proton hide

"Helicity sum rule:"

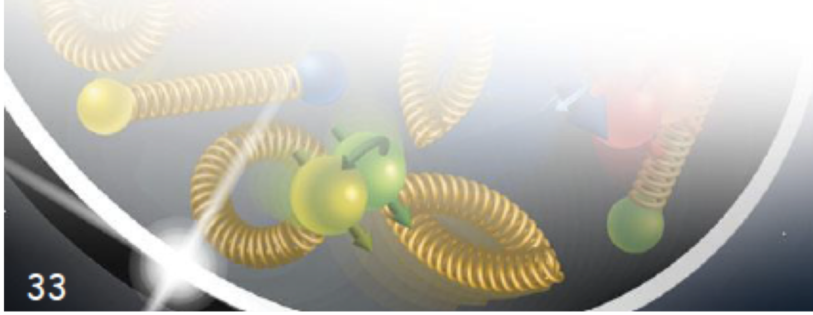
$$\frac{1}{2}\hbar = \left\langle P, \frac{1}{2} \middle| J^z_{QCD} \middle| P, \frac{1}{2} \right\rangle = \overbrace{\sum_q \frac{1}{2} S^z_q}^{\text{total quark spin}} + \overbrace{S^z_g}^{\text{gluon spin}} + \sum_q \overbrace{L^z_q + L^z_g}^{\text{angular momentum}}$$







1/2 - Gluon - Quarks = orbital angular momentum



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