Finite Energy Sum Rules For $\pi p \rightarrow \pi \eta p$
In PROGRESS

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Exotic Spectroscopy

Mesons: $q\bar{q}$

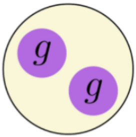
$P = (-1)^{L+1}$

$C = (-1)^{L+S}$

$J^{PC}= 0^{-+}, 0^{++}, 1^{--}, 1^{+-}, 1^{++}, 2^{--}, 2^{-+}, 2^{++}, \ldots$

Exotic

$0^{--}, 0^{+-}, 1^{-+}, 2^{+-}, \ldots$



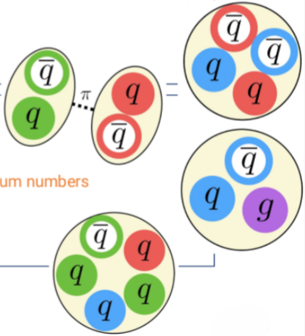
glueballs or hybrid mesons or multi-quark states or molecules

Baryons: qqq

Observation is difficult:

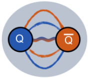
- ‘exotics’ hide in plain sight since they have the same quantum numbers
- Large masses

Only structure to distinguish them



0^{--}	0^{-+}	0^{+-}	0^{++}
1^{--}	1^{-+}	1^{+-}	1^{++}
2^{--}	2^{-+}	2^{+-}	2^{++}
3^{--}	3^{-+}	3^{+-}	3^{++}
$\vdots$	$\vdots$	$\vdots$	$\vdots$

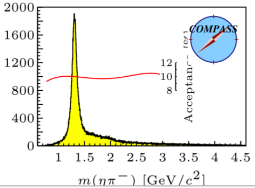
Exotic mesons

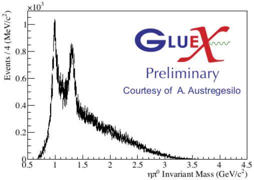


The lightest is π_1 $\quad J^{PC} = 1^{-+}$

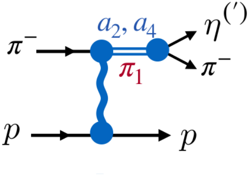
Decay mode $\quad \pi_1 \to \pi\eta$

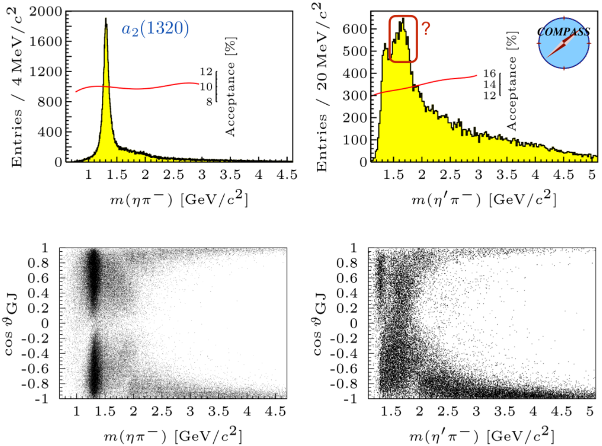
$$1^{-+} = (0^{-+} \otimes 0^{-+})_{P\text{-wave}}$$





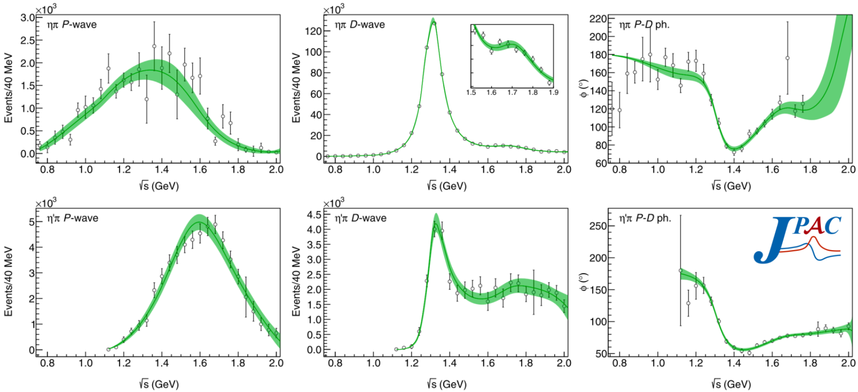
$\eta\pi$ at COMPASS COMPASS PLB740 (2015)





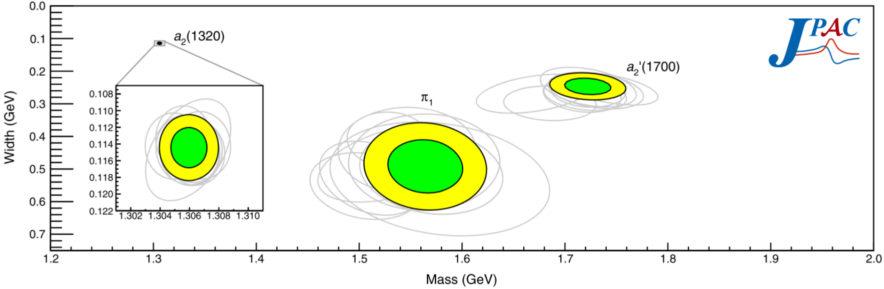
Low Energy Fit of P and D waves Rodas et al PRL122 (2019)

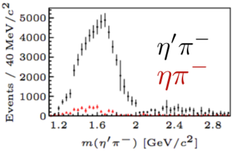
$\pi_1(1400)$ vs $\pi_1(1600)$

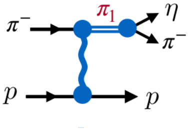


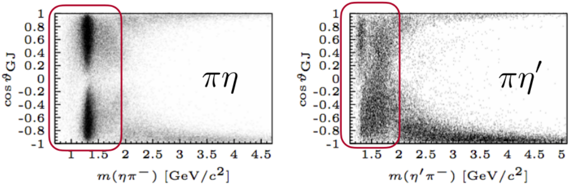
Low Energy Fit of P and D waves Rodas et al PRL122 (2019)

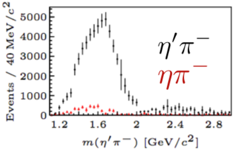
Poles	Mass (MeV)	Width (MeV)
$a_2(1320)$	$1306.0 \pm 0.8 \pm 1.3$	$114.4 \pm 1.6 \pm 0.0$
$a_2'(1700)$	$1722 \pm 15 \pm 67$	$247 \pm 17 \pm 63$
π_1	$1564 \pm 24 \pm 86$	$492 \pm 54 \pm 102$

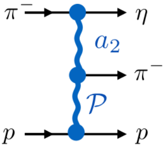



Events / 40 MeV/c²
5000
4000
3000
2000
1000
0
1.2
1.6
2
2.4
2.8
m(η′π⁻) [GeV/c²]
η′π⁻
ηπ⁻


π⁻
π₁
η
π⁻
p
p

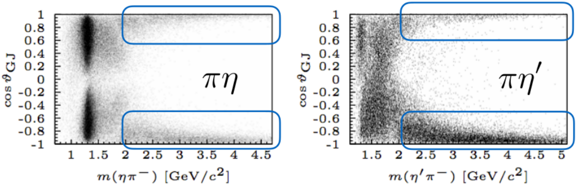

cos ϑGJ
πη
m(ηπ⁻) [GeV/c²]
πη′
m(η′π⁻) [GeV/c²]





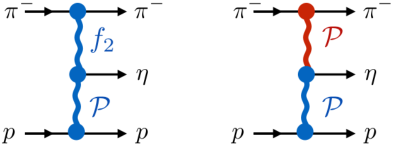
$\cos\theta_{GF} \sim 1 \rightarrow \eta$ forward

$\cos\theta_{GF} \sim -1 \rightarrow \eta$ backward



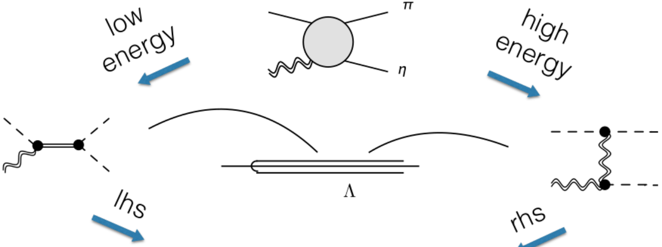
Exotic meson related to
Forward-backward asymmetry

Asymmetry related to
even-odd waves interferences



Finite Energy Sum rules

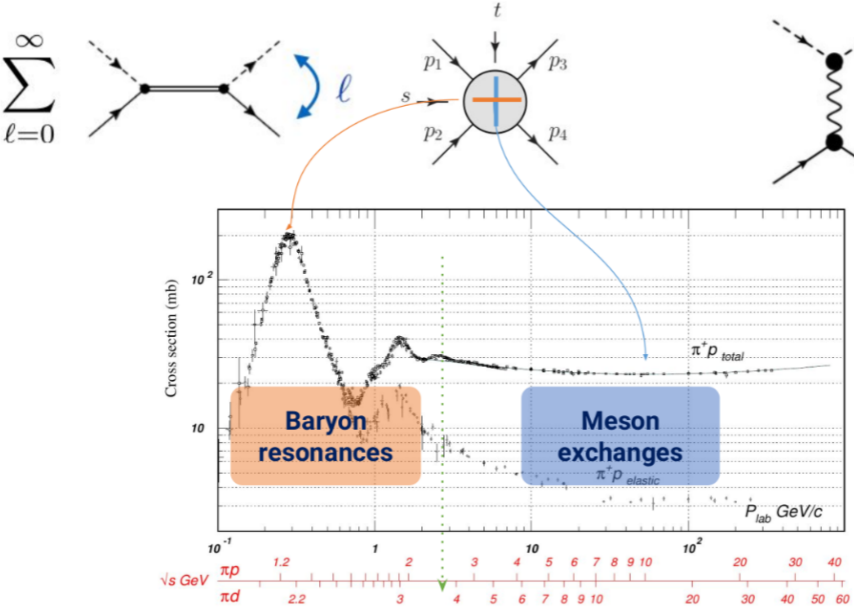
$$A(s_1, t_1, t_2, s_2/s) = \sum_i A_i(s_1, t_1, t_2) \left(\frac{s_2}{s}\right)^2$$



at fixed t_1,t_2 $$\int^{\Lambda} ImA_i(s_1, t_1, t_2) = \int_{\Lambda} ImA_i(s_1, t_1, t_2)$$

- Derived using Cauchy's theorem: $\oint_C A(s,t)ds = 0$:
- Connect low-energy and high-energy dynamics.
- Predict high-energy observables from low-energy data.
- Constrain low-energy models using high-energy results.

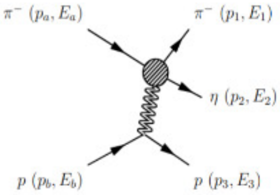
Finite Energy Sum rules

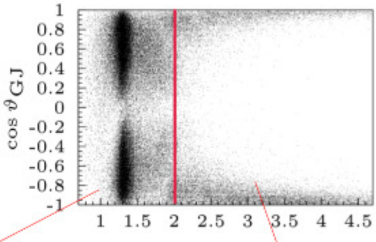


Summary: Why Study $\pi p \to \pi\eta p$?

- High-quality data available from the COMPASS experiment.
- Theoretical Motivation:
  - Test the analytic structure and consistency of scattering amplitudes.
  - Apply Finite Energy Sum Rules (FESR) for the **first time** to a $2 \to 3$ process; a novel theoretical challenge.
- Phenomenological Goals:
  - Connect resonance-region dynamics to high-energy behavior through FESR.
  - Constrain resonance parameters using high-energy data potentially shed light on exotic candidates like the π_1.
- Applications:
  - Improve modeling of vertex functions in double-Regge exchange frameworks.
  - Reduce uncertainty in the extraction of the π_1 pole position (see PRL 122, 042002 (2019)).

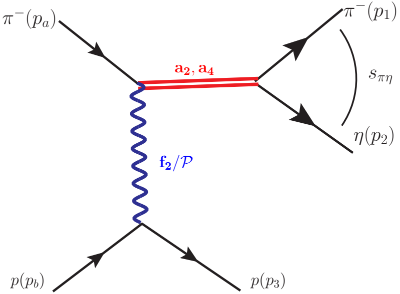
Analysis of $\pi p \to \pi \eta p$





Low Energy Limit: Resonance production, Single Regge exchange

High Energy Limit : Double Regge exchange



At low energies, the amplitude is primarily dominated by resonance production and can be expressed as a sum of partial wave amplitudes, which can be extracted from experimental data:

$$A^{PW}(s_{\eta\pi}, \Omega) = \sum_{l=1}^{6} a_{l,1}(s_{\eta\pi}) \sin(\Phi)\, Y_l^1(\theta, 0) + a_{2,2}(s_{\eta\pi}) \sin(2\Phi)\, Y_2^2(\theta, 0)$$

SRL Amplitude

To extract the partial wave amplitudes from data, we use the intensities and relative phases provided in the COMPASS analysis (Phys. Lett., B740:303-311, 2015). The partial waves are written as

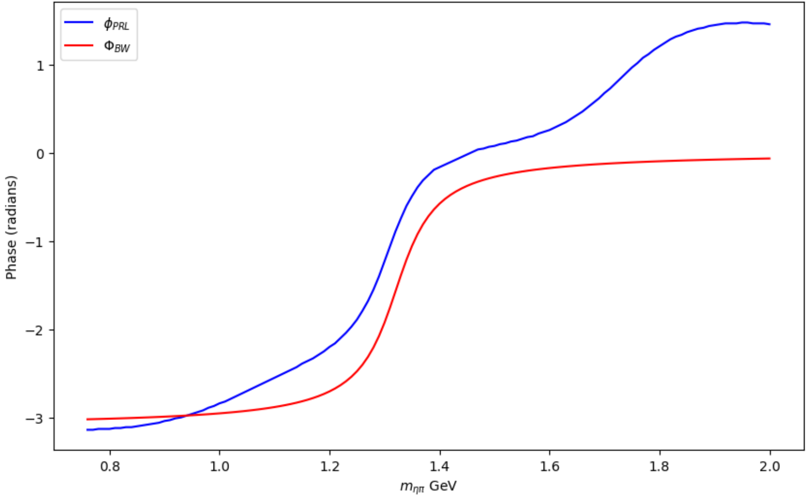
$$a_{l,M}(s_{\eta\pi}) = \sqrt{\frac{I_{l,M}(s_{\eta\pi})}{k}}\, e^{i\delta_{l,M}(s_{\eta\pi})}$$

where $I_{l,M}$ and $\delta_{l,M}$ are the experimental intensity and phases relative to the $l = 2,\ M = 1$ wave i.e.$\delta_{l,M} = \phi_{l,M} - \phi_{2,1}$:

$$k(s_{\eta\pi}) = \frac{\lambda^{1/2}(s_{\eta\pi}, m_\eta^2, m_\pi^2)}{2\sqrt{s_{\eta\pi}}}$$

is the breakup momentum factor.

SRL Amplitude



The $l = 2, M = 1$ wave phase calculated from the Breit Wigner of a_2 meson vs. the one calculated from the theoretical results of PRL model PRL 122, 042002 (2019)

SRL Amplitude

To obtain an analytic amplitude, we remove the dynamical singularities by dividing out the kinematic factor:

$$K = 4\sqrt{s_{\eta\pi}}\,|p_2|\,|q|\,|p_a|\sin\epsilon\sin\theta\sin\phi$$

In the single Regge limit, $s, s_{\eta p} \to \infty$ with fixed $\kappa = s_{\eta p}/s$, the kinematic factor simplifies to:

$$4\sqrt{s_{\eta\pi}}\,|q|\,|p_a|\sin\epsilon \simeq 2s\sqrt{-u_p}, \qquad K \simeq 2s\sqrt{-u_p}\cdot|p_2|\sin\theta\sin\phi$$

The reduced partial wave amplitude, obtained after dividing out K, reads:

$$\widehat{A}^{PW}(s_{\eta\pi},\Omega) = \frac{A^{PW}}{K} = \frac{1}{2s\sqrt{-u_p}}\left[\sum_{\ell=1}^{6}\frac{a_{\ell,1}}{|p_2|}\frac{Y_\ell^1(\theta,0)}{\sin\theta} + \frac{1}{2}\sqrt{\frac{15}{2\pi}}\frac{a_{2,2}}{|p_2|}\sin\theta\cos\phi\right]$$

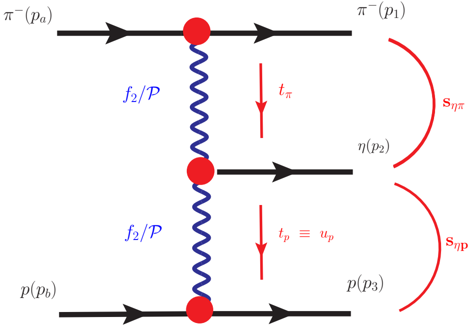
The generic amplitude for the double Regge exchange can be expressed as in arXiv:2104.10646v2 [hep-ph] 19 Jul 2021 :

$$T(\alpha_1, \alpha_2; s_{\pi\eta}, s_{\eta p}) = K\,\Gamma(1-\alpha_1)\,\Gamma(1-\alpha_2)\frac{(\alpha' s_{\pi\eta})^{\alpha_1}(\alpha' s_{\eta p})^{\alpha_2}}{\alpha' s}$$
$$\left[\frac{\xi_1\,\xi_{21}}{\kappa^{\alpha_1}}\,V(\alpha_1,\alpha_2,\kappa) + \frac{\xi_2\,\xi_{12}}{\kappa^{\alpha_2}}\,V(\alpha_2,\alpha_1,\kappa)\right]$$

where

$$V(\alpha_1,\alpha_2,\kappa) = \frac{\Gamma(\alpha_1-\alpha_2)}{\Gamma(1-\alpha_2)}\,{}_1F_1\left(1-\alpha_1, 1-\alpha_1+\alpha_2, -\kappa\right)$$
$$\kappa = \frac{\alpha' s_{\pi\eta} s_{\eta p}}{s}$$
$$\xi_n = \frac{1+e^{-i\pi\alpha_n}}{2}$$
$$\xi_{nm} = \frac{1+e^{-i\pi(\alpha_n-\alpha_m)}}{2}$$

DRL Amplitude- Fast pion



The total amplitude $A(s_{\eta\pi}, \Omega)$ is the sum of four possible double-Regge amplitudes

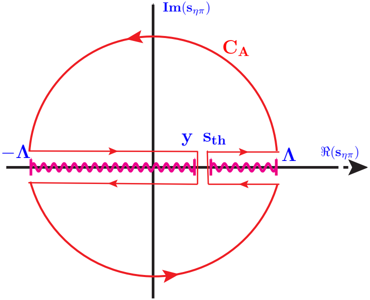
$$A(s_{\eta\pi}, \Omega) = c_{f_2\mathbb{P}} T_{f_2\mathbb{P}} + c_{f_2f_2} T_{f_2f_2} + c_{\mathbb{P}\mathbb{P}} T_{\mathbb{P}\mathbb{P}} + c_{\mathbb{P}f_2} T_{\mathbb{P}f_2}$$

Where c were fitted to the data in arXiv:2104.10646v2 [hep-ph] 19 Jul 2021, with

$$\alpha_{f_2}(t) = 0.47 + 0.89\,t$$
$$\alpha_{\mathbb{P}}(t) = 1.08 + 0.25\,t$$

We define the total amplitude $\hat{A}$ for the **fast pion** case to be:

$$\widehat{A}^{DRL}(s_{\eta\pi}, \Omega) = \frac{A(s_{\eta\pi}, \Omega)}{K} = \frac{\sum_{i=1}^{4} c_i T^i}{K} = \sum_{i=1}^{4} c_i \widehat{T}^i$$



- RHC for $s_{\eta\pi} > (m_\eta + m_\pi)^2$ (physical threshold)
- LHC for $s_{\eta\pi} < 0$, arising from crossed-channel (u-channel) processes, starts at $t_\eta = (m_\pi + m_\eta)^2$

$$y = m_\pi^2 + 2\, m_\pi\, m_\eta + u_p - t_\pi$$

This leads to the following finite-energy relation:

$$2i \int_{-\Lambda}^{y} \mathrm{Im}\, \widehat{A}^{\mathrm{SRL}}\, ds_{\eta\pi} + 2i \int_{s_{\mathrm{th}}}^{\Lambda} \mathrm{Im}\, \widehat{A}^{\mathrm{SRL}}\, ds_{\eta\pi} = \int_{C_\Lambda} \widehat{A}^{\mathrm{DRL}}\, ds_{\eta\pi}$$

The left hand side of the FESR is:

$$LHS = 2\,i \int_{s_{th}}^{\Lambda} ds_{\eta\pi} \frac{1}{2s\sqrt{-u_p}} \left[\sum_{\ell=1}^{6} \frac{\text{Im}\, a_{\ell,1}(s_{\eta\pi})}{|p_\eta|} \frac{Y_\ell^1(\theta,0)}{\sin\theta} + \frac{1}{2}\sqrt{\frac{15}{2\pi}} \frac{\text{Im} a_{2,2}(s_{\eta\pi})}{|p_\eta|} \sin\theta\cos\phi \right]$$

$$+ 2\,i \int_{-\Lambda}^{y} ds_{\eta\pi} \frac{1}{2s\sqrt{-u_p}} \left[\sum_{\ell=1}^{6} \frac{\text{Im} a_{\ell,1}(t_\eta)}{|p_\eta|} \frac{Y_\ell^1(\theta,0)}{\sin\theta} + \frac{1}{2}\sqrt{\frac{15}{2\pi}} \frac{\text{Im} a_{2,2}(t_\eta)}{|p_\eta|} \sin\theta\cos\phi \right]$$

The right-hand side of the FESR is given by the closed contour integral over a circle of radius Λ :

$$RHS = \sum_i c^i \oint \widehat{T}_i^{DRL} ds_{\eta\pi} = \sum_i c^i \Bigg[-2\pi\, i[\xi_2(\alpha' s)^{\alpha_2-1}] \sum_{j=0}^{\infty} \frac{[(-1)^j + 1](\alpha'\Lambda)^{\alpha_1-\alpha_2+1}}{j!}$$

$$\frac{\Gamma(1-\alpha_2+j)}{\Gamma(2-\alpha_2+\alpha_1+j))} \left(\frac{\alpha'\Lambda s_{\eta p}}{s}\right)^j \Bigg]$$

FESR-Lowest Moment: $j = 0$

We begin by applying the Finite Energy Sum Rules (FESR) to the lowest moment, $j = 0$, which corresponds to the ϕ-independent term. We fix $u_p = -0.2$ GeV^2, and consider three different cutoff values:

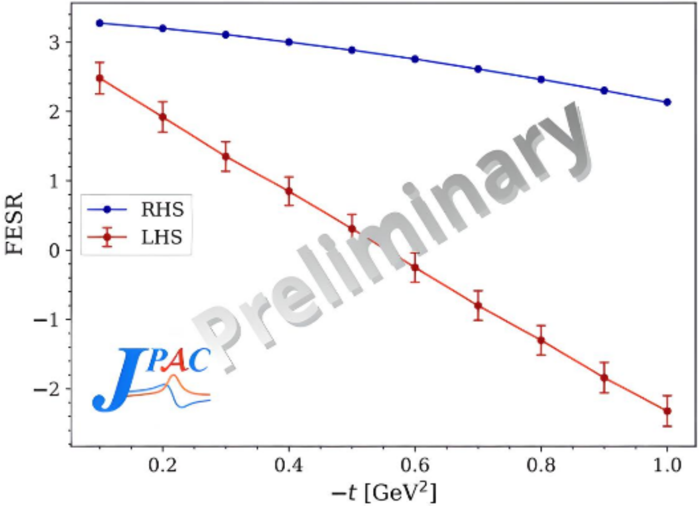
- $\Lambda = 1.6^2 = 2.56$ GeV^2: includes partial waves up to $\ell = 3$.
- $\Lambda = 4$ GeV^2: includes partial waves up to $\ell = 4$.
- $\Lambda = 2.96^2 = 5.76$ GeV^2: includes all partial waves, i.e., $\ell = 1, \ldots, 6$.

Channel	Parameter	MIN	MIN+$f/\mathbb{P}$	MIN+$\mathbb{P}/\mathbb{P}$
$\eta\pi$	$c_{f_2\mathbb{P}}$	—	-0.20	—
	$c_{f_2 f_2}$	-11.82	-8.99	-10.86
	$c_{\mathbb{P}\mathbb{P}}$	—	—	0.0073

Table: Fitted parameters for the DRL amplitude from arXiv:2104.10646v2 [hep-ph] 19 Jul 2021

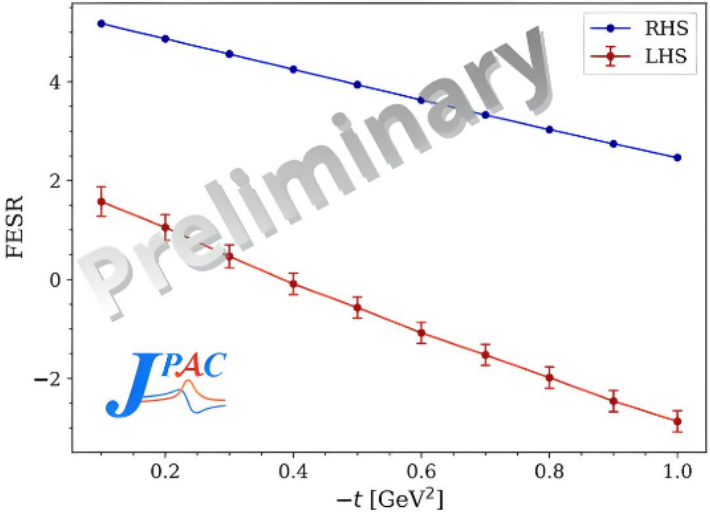
FESR-Zeroth Moment Results

$\Lambda = 2.56$ GeV^2:



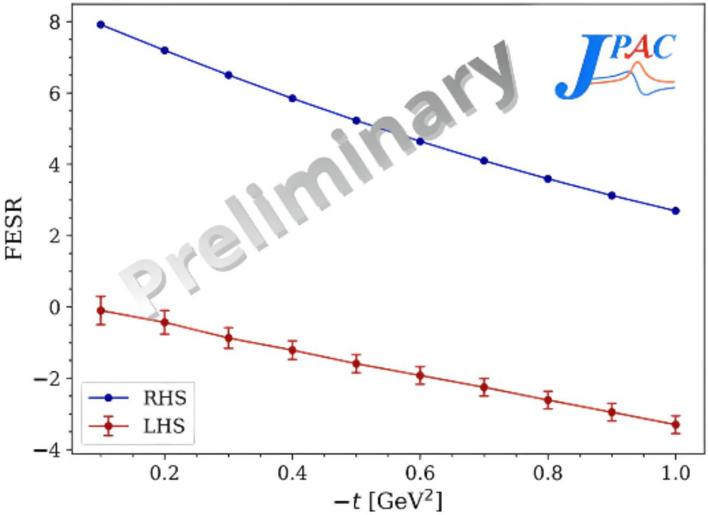
FESR-Zeroth Moment Results

$\Lambda = 4\ \text{GeV}^2$



FESR-Zeroth Moment Results

$\Lambda = 5.76\ \text{GeV}^2$:



Summary and Outlook

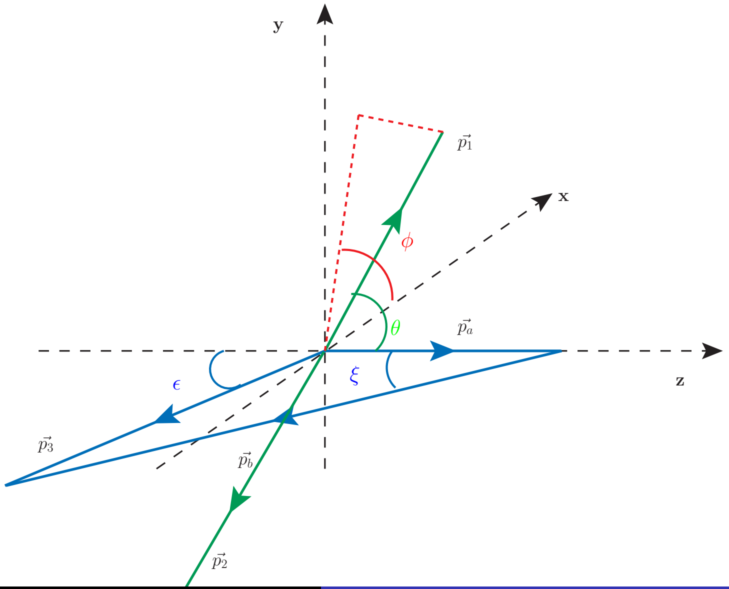
- Demonstrated the importance of Finite Energy Sum Rules (FESR) for understanding the dynamics of $\pi^- p \to \pi^- \eta p$.
- Presented the first theoretical derivation of both the LHS (low-energy) and RHS (high-energy) of FESR for a $2 \to 3$ process (fast pion limit).
- Applied the zeroth moment ($j = 0$) using different cutoffs and partial wave sets - best agreement at $\Lambda = 4$ GeV^2, but LHS and RHS differ by a factor ≈ 2.
- Increasing the number of partial waves did not resolve the discrepancy; its origin remains under investigation.
- Currently exploring higher moments to probe consistency and identify dominant contributions.
- This is the first step - FESR is well-tested in $2 \to 2$ processes, but its application to $2 \to 3$ is novel and requires further theoretical and numerical work.


Thank you

Back-Up Slides

Kinematics

12-GJ-frame:

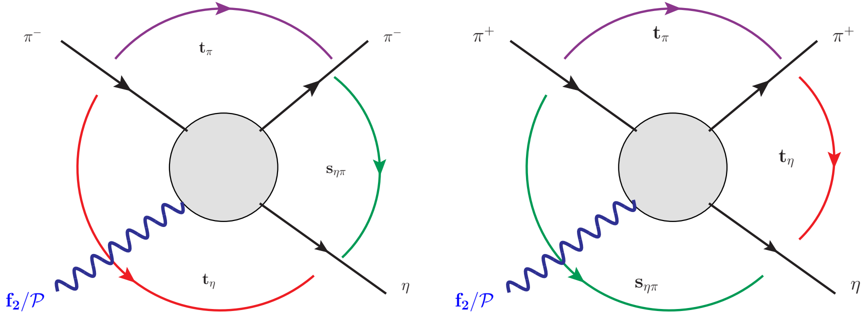


Kinematics

$$
\begin{aligned}
\boldsymbol{p}_a &= |\boldsymbol{p}_a|(0, 0, 1) \\
\boldsymbol{p}_b &= |\boldsymbol{p}_b|(-\sin\xi, 0, -\cos\xi) \\
\boldsymbol{p}_3 &= |\boldsymbol{p}_3|(-\sin\epsilon, 0, -\cos\epsilon) \\
\boldsymbol{p}_1 &= |\boldsymbol{p}_1|(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \\
\boldsymbol{p}_2 &= -|\boldsymbol{p}_1|(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)
\end{aligned}
$$

$$
s = (p_a + p_b)^2\,, \qquad t_i = (p_a - p_i)^2\,, \quad u_i = (p_b - p_i)^2\,, \qquad s_{ij} = (p_i + p_j)^2\,.
$$

Crossed-Channels



(a) "s-channel" (b) "u-channel"

Figure: Representation of the crossed channels for $\pi p \to \pi\eta p$