Towards three-pion amplitudes from lattice QCD

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July 29th, 2025

based on work with…
Athari Alotaibi Raúl Briceño Robert Edwards Andrew Jackura
Fernando Romero-López Steve Sharpe Christopher Thomas

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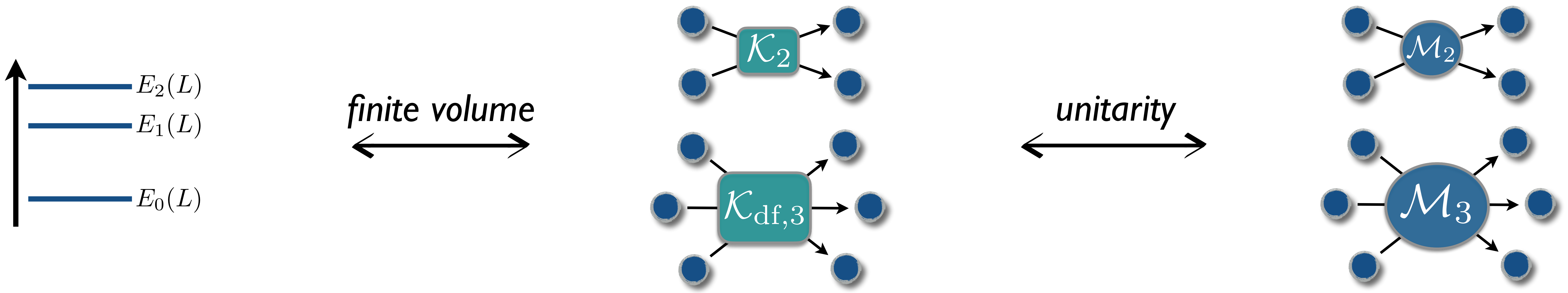
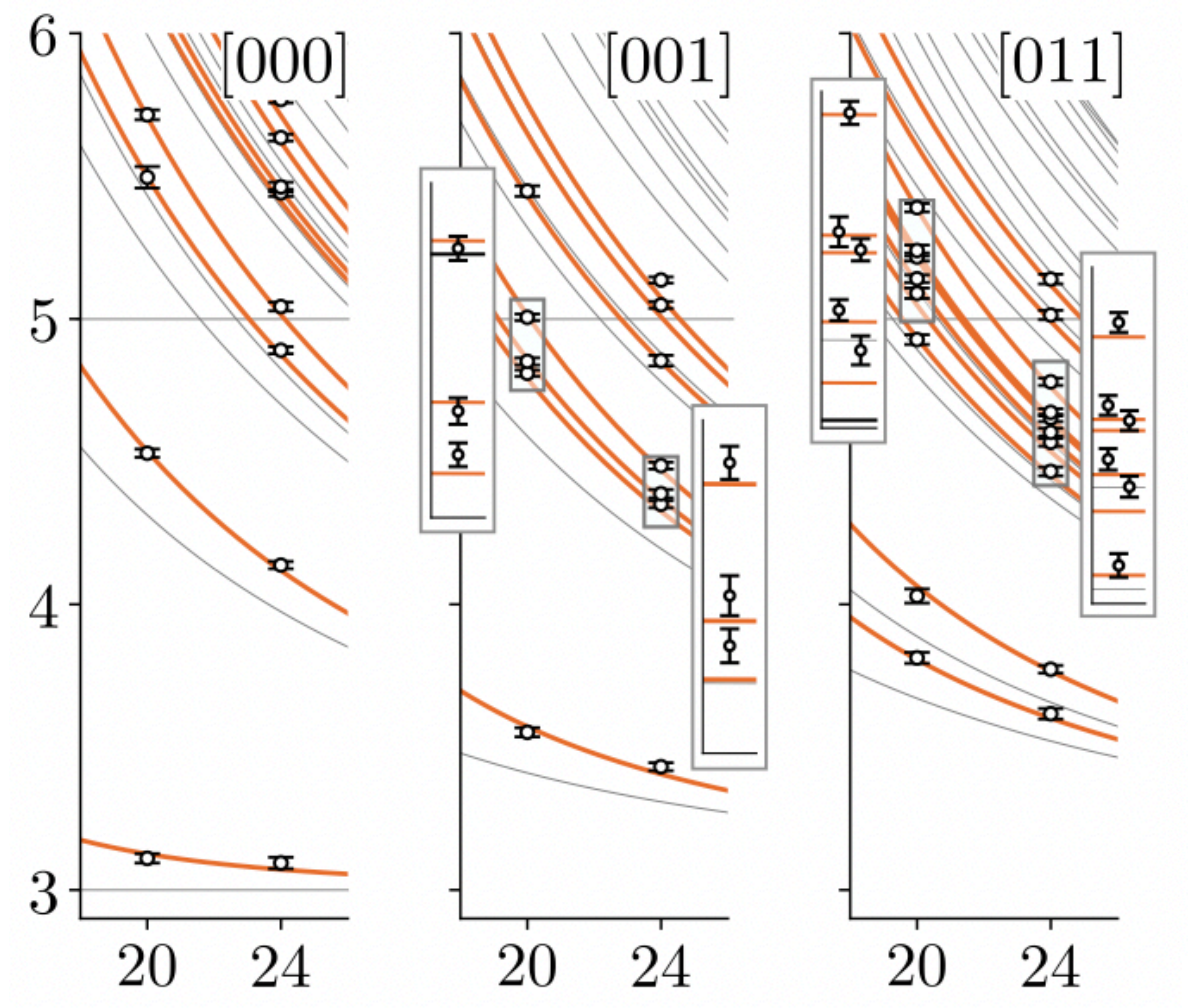
From the finite-volume energies to amplitudes

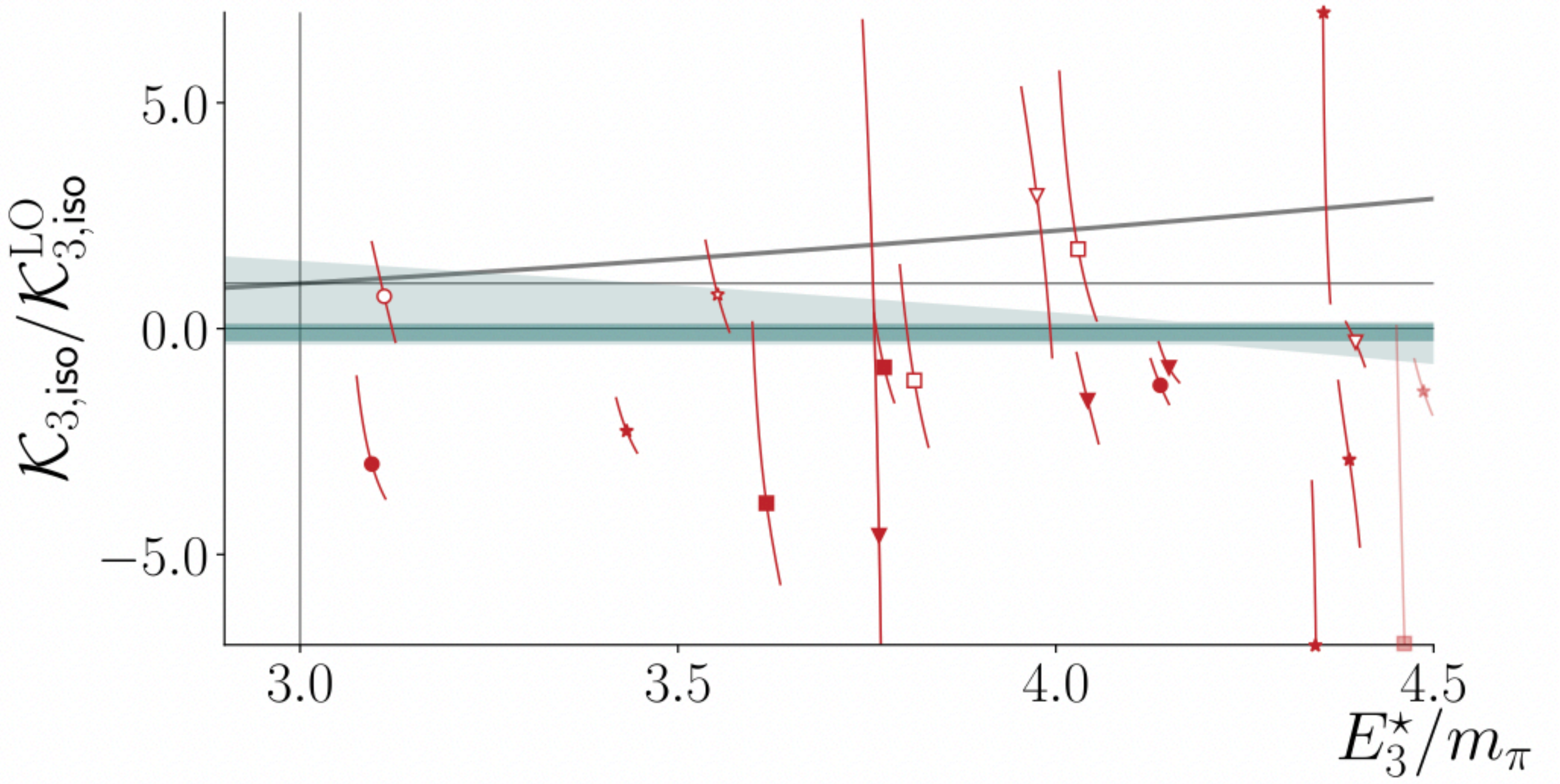


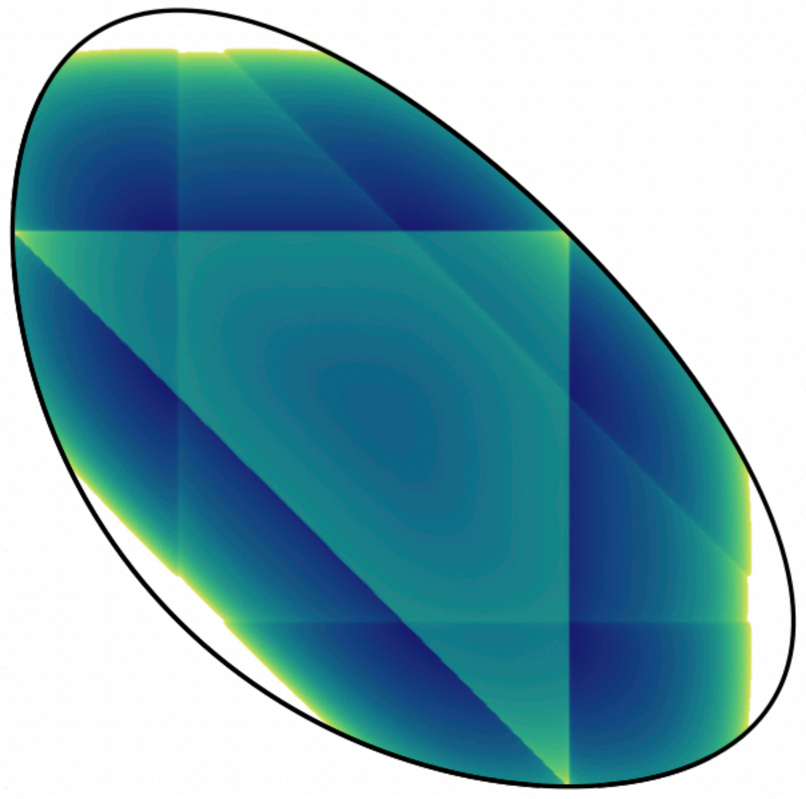
finite volume

unitarity

MTH and Sharpe (2014-2016)





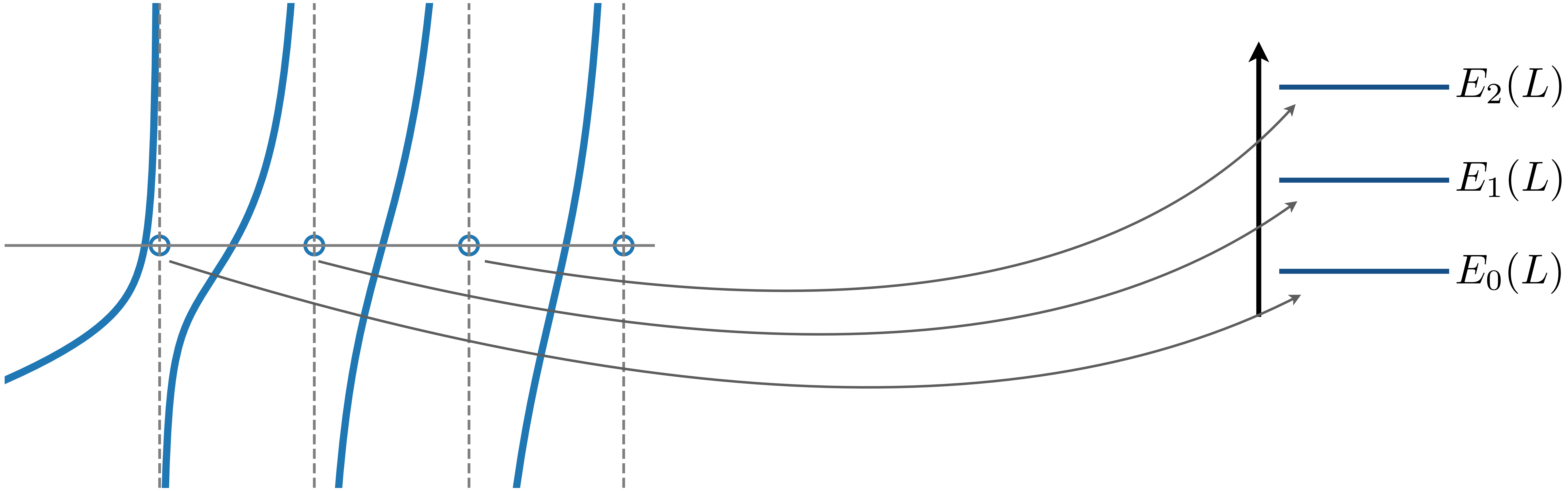


MTH, Briceño, Edwards, Thomas, Wilson, (2020)

RFT finite-volume formalism: *Derivation*

RFT = relativistic field theory

Poles of finite-volume correlator $(C_L(E))$ = *finite-volume energies*

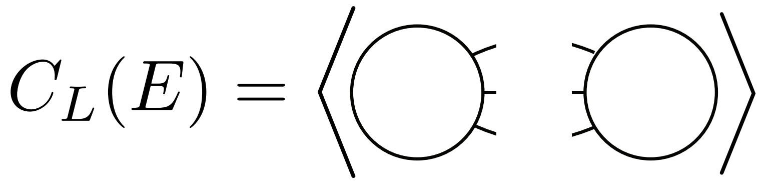


RFT finite-volume formalism: *Derivation*

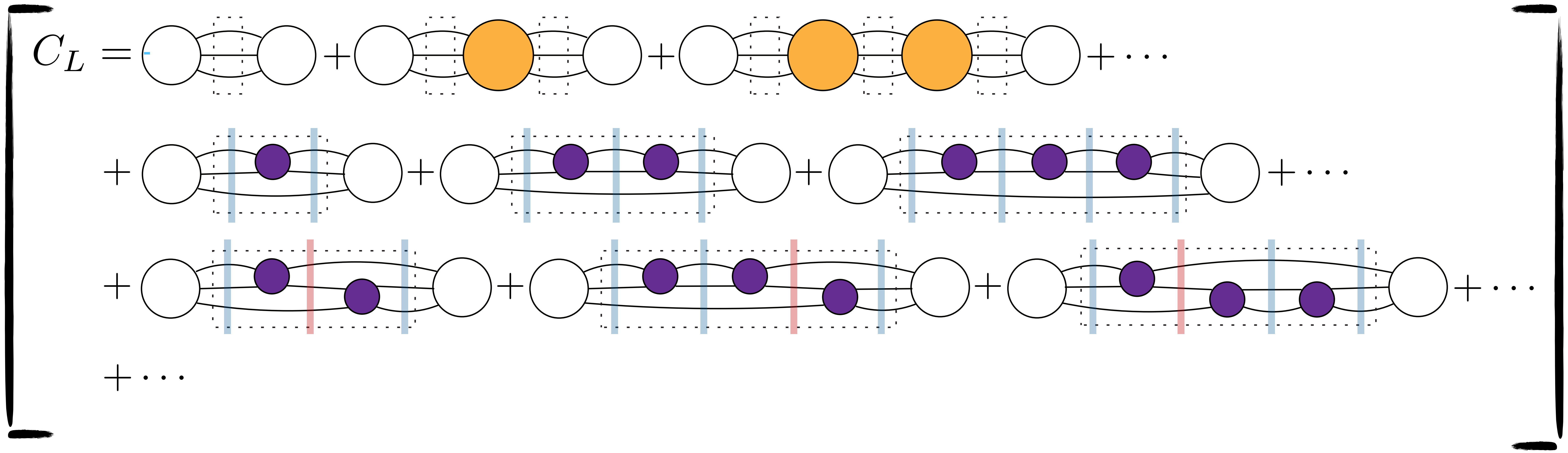
RFT = relativistic field theory

Poles of finite-volume correlator ($C_L(E)$) = *finite-volume energies*

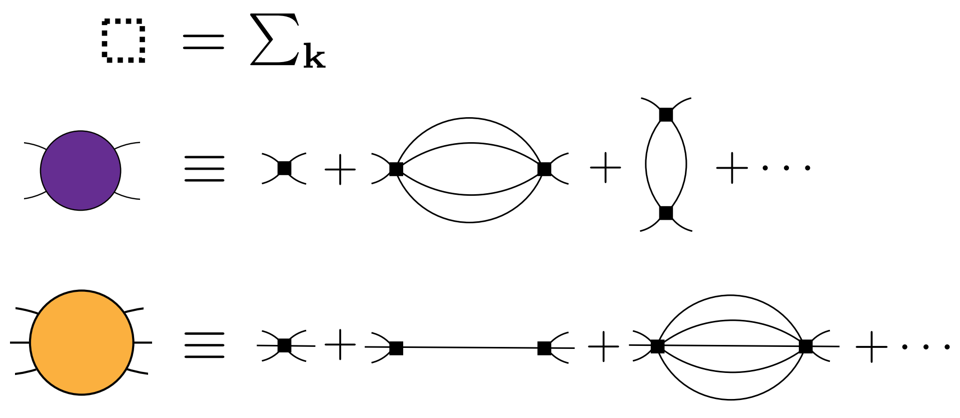
$$\textit{Poles inside}\left[\sum \textit{Feynman diagrams for } C_L(E)\right] = E_n(L)$$



Poles inside



$= E_n(L)$

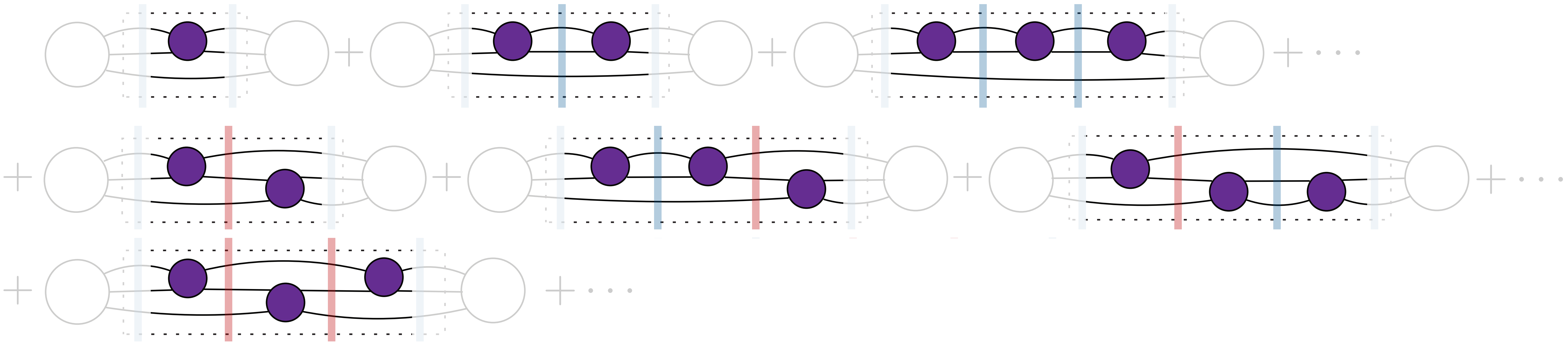


MTH, Sharpe (2014-2016)

See also Döring, Mai, Hammer, Pang, Rusetsky

RFT finite-volume formalism: *Derivation*

RFT = relativistic field theory

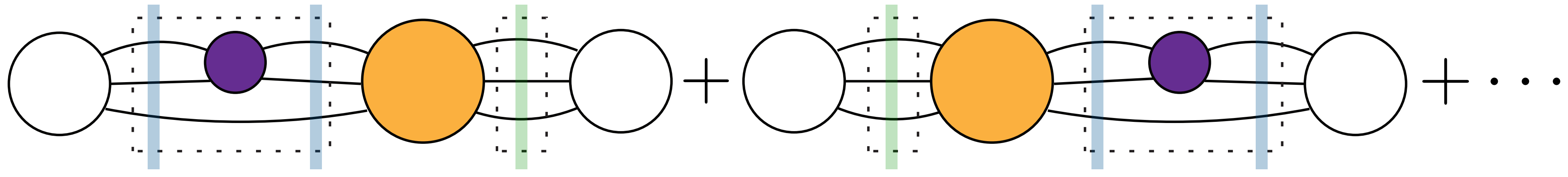


$$\approx \mathbf{K}_2 \sum_{n=0}^{\infty} [(\mathbf{F}+\mathbf{G})\mathbf{K}_2]^n = \mathbf{K}_2 \frac{1}{1-(\mathbf{F}+\mathbf{G})\mathbf{K}_2}$$

Significant effort to… make the meaning of the cuts precise

understand how each cut splits into "smooth bit" + "singular bit"

understand how to re-sum "smooth bits"



Keeping careful track of how the series starts gives…

$$\mathbf{F}_3 = \frac{\mathbf{F}}{3} + \mathbf{F}\mathbf{K}_2\left[1-(\mathbf{F}+\mathbf{G})\mathbf{K}_2\right]^{-1}\mathbf{F}$$

Putting in the short-distance 3-to-3 gives

$$C_L(P) - C_\infty(P) = \mathbf{A}\mathbf{F}_3 \frac{1}{1-\mathbf{K}_{\mathrm{df},3}\mathbf{F}_3}\mathbf{A}'$$

MTH, Sharpe (2014-2016) *See also Döring, Mai, Hammer, Pang, Rusetsky*

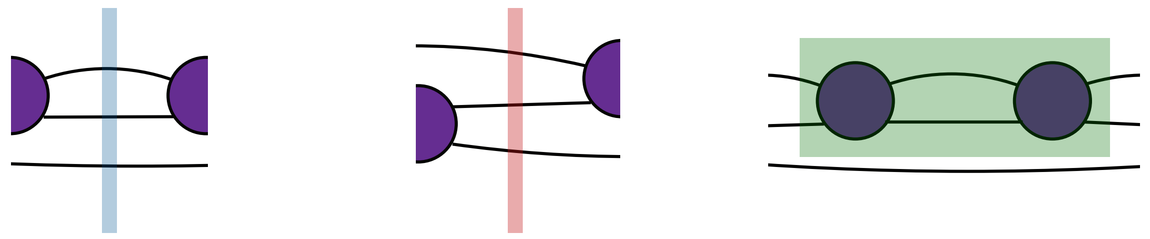
RFT quantization condition

RFT = relativistic field theory

$$\det\left[\mathbf{K}_{\mathsf{df},3}^{-1}(E_{\mathsf{cm}}) - \mathbf{F}_3(E,\boldsymbol{P},L)\right] = 0$$

$\mathbf{F}_3(E,\boldsymbol{P},L) =$ Matrix of functions depending on kinematics + two-particle dynamics

$$\mathbf{F}_3 = \frac{\mathbf{F}}{3} + \mathbf{F}\mathbf{K}_2\left[1 - (\mathbf{F}+\mathbf{G})\mathbf{K}_2\right]^{-1}\mathbf{F}$$



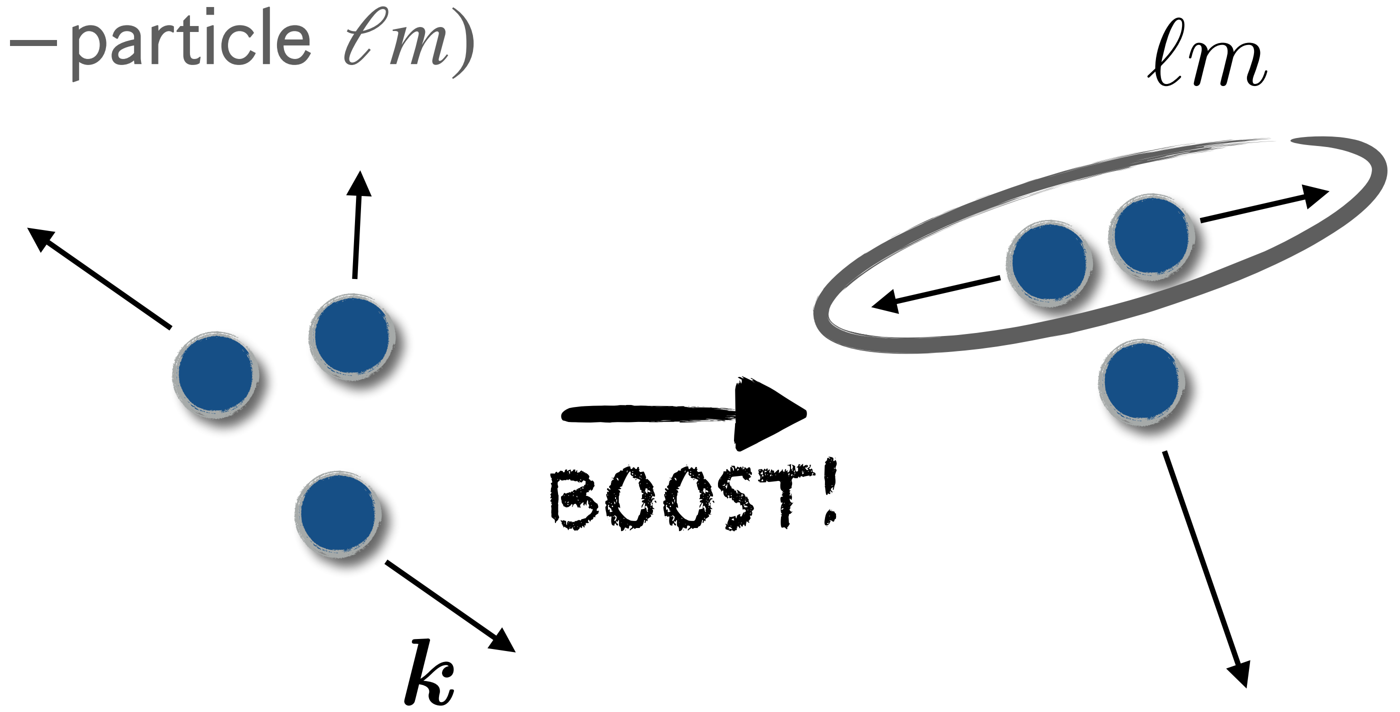
Matrices on tensor-product space: $(\text{spectator flavor space}) \otimes (\text{spectator } \boldsymbol{k} \in \frac{2\pi}{L}\mathbb{Z}^3 \text{ space}) \otimes (\text{two}-\text{particle } \ell m)$

Holds only for three-particle energies

Neglects $e^{-M_\pi L}$

Requires sub-threshold continuation of $\mathbf{K}_2$

Scheme-dependent $\mathbf{K}_{\mathrm{df},3}$ related to physical amplitude via known on-shell integral equations

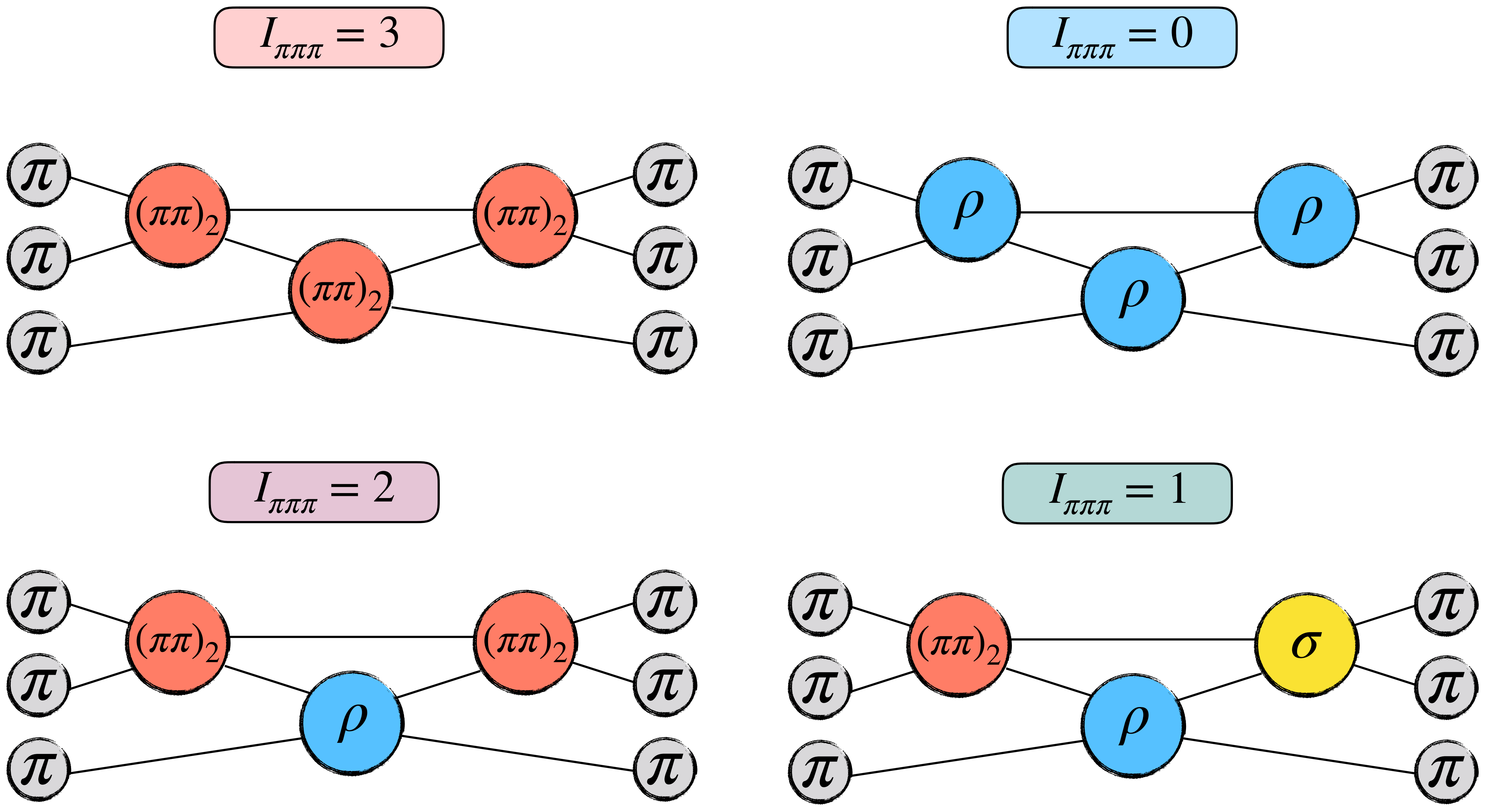


MTH, Sharpe (2014-2016)

Three pions with isospin

- ☐ Four possible iso-spin channels for three pions

$$1 \otimes 1 \otimes 1 = (0 \oplus 1 \oplus 2) \otimes 1 = \underbrace{1}_{I_{\pi\pi} = 0} \oplus \underbrace{(0 \oplus 1 \oplus 2)}_{I_{\pi\pi} = 1} \oplus \underbrace{(1 \oplus 2 \oplus 3)}_{I_{\pi\pi} = 2}$$



- ☐ Four quantization conditions

$$I_{\pi\pi\pi} = 0 \qquad \begin{array}{cc} (\rho) & \\ \left(\square\right) & (\rho) \end{array}$$

$$I_{\pi\pi\pi} = 1 \qquad \begin{array}{cc} \begin{matrix} (\sigma) & (\rho) & (\pi\pi)_2 \end{matrix} & \\ \begin{pmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{pmatrix} & \begin{matrix} (\sigma) \\ (\rho) \\ (\pi\pi)_2 \end{matrix} \end{array}$$

$$I_{\pi\pi\pi} = 2 \qquad \begin{array}{cc} \begin{matrix} (\rho) & (\pi\pi)_2 \end{matrix} & \\ \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix} & \begin{matrix} (\rho) \\ (\pi\pi)_2 \end{matrix} \end{array}$$

$$I_{\pi\pi\pi} = 3 \qquad \begin{array}{cc} (\pi\pi)_2 & \\ \left(\square\right) & (\pi\pi)_2 \end{array}$$

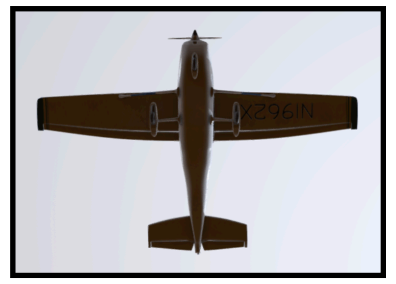
MTH, Romero-López Sharpe, JHEP (2020)

Finite-volume group theory

- Focusing on a cubic box → total momentum determines symmetry group

$\boldsymbol{n}_P$	group	dim	$n_{\text{1d irreps}}$	$n_{\text{2d irreps}}$	$n_{\text{3d irreps}}$	irrep names(dim)
$[000]$	O_h	48	4	2	4	$A_1^+(1), A_2^+(1), E^+(2), T_1^+(3), T_2^+(3), (+ \to -)$
$[001]$	C_4	8	4	1		$A_1(1), A_2(1), B_1(1), B_2(1), E_2(2)$
$[011]$	C_2	4	4			$A_1(1), A_2(1), B_1(1), B_2(1)$
$[111]$	C_3	6	2	1		$A_1(1), A_2(1), E_2(2)$

- For any object one can identify a linear combination of rotations that transforms as given irrep, row (Γ, μ)

$\Bigg($  $\Bigg)_{A_1^+} = 1\times$  $+1\times$  $+1\times$  $+1\times$  $+\cdots$

- Numerically determining a representation of the rotation space → row-specific projectors

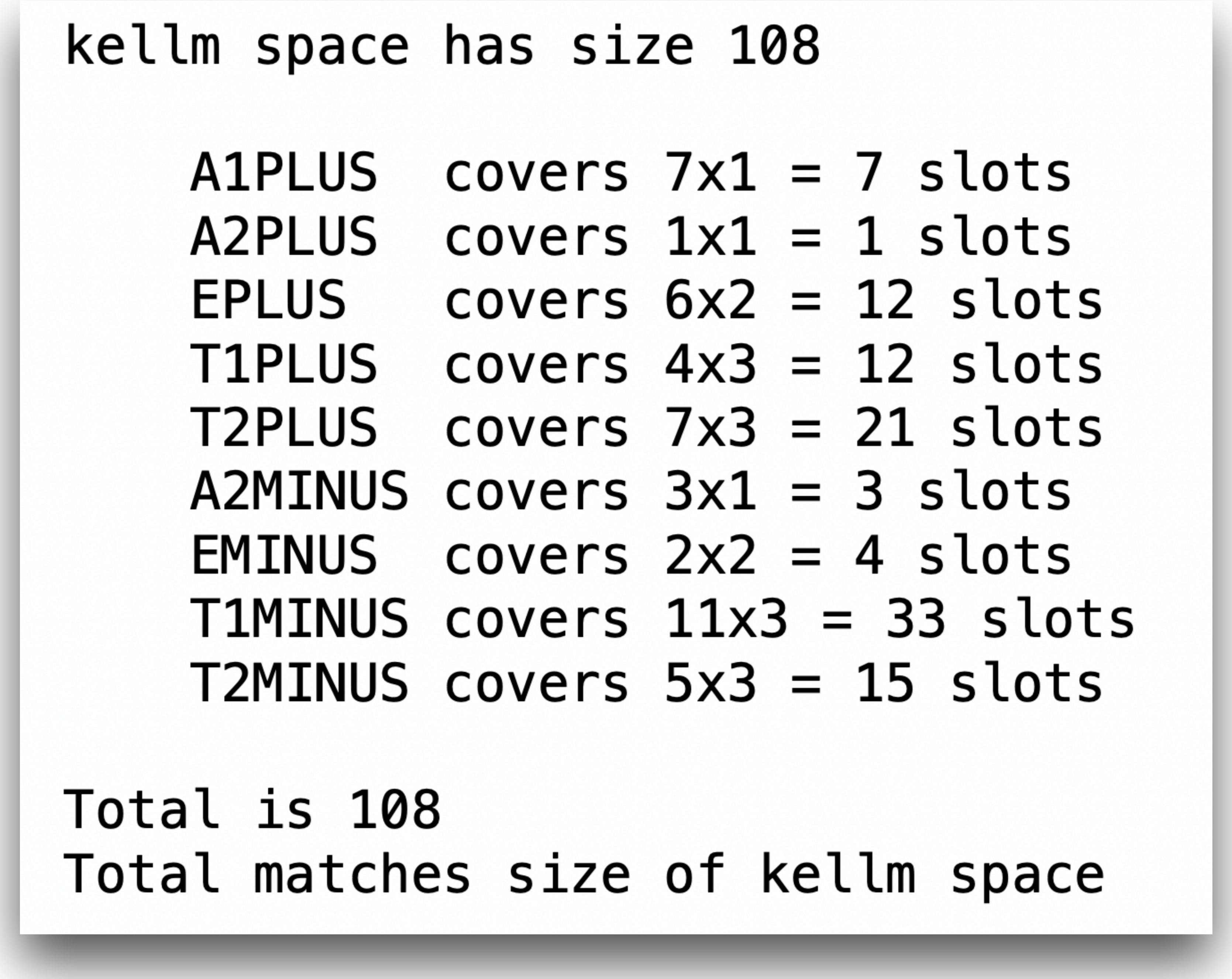
Subducing the index space

☐ Returning to the $I_{\pi\pi\pi} = 2$ space

$$(\text{spectator flavor space}) \otimes (\text{spectator } \boldsymbol{k} \in \frac{2\pi}{L}\mathbb{Z}^3 \text{ space}) \otimes (\text{two}-\text{particle } \ell m)$$

108 x 108 matrices

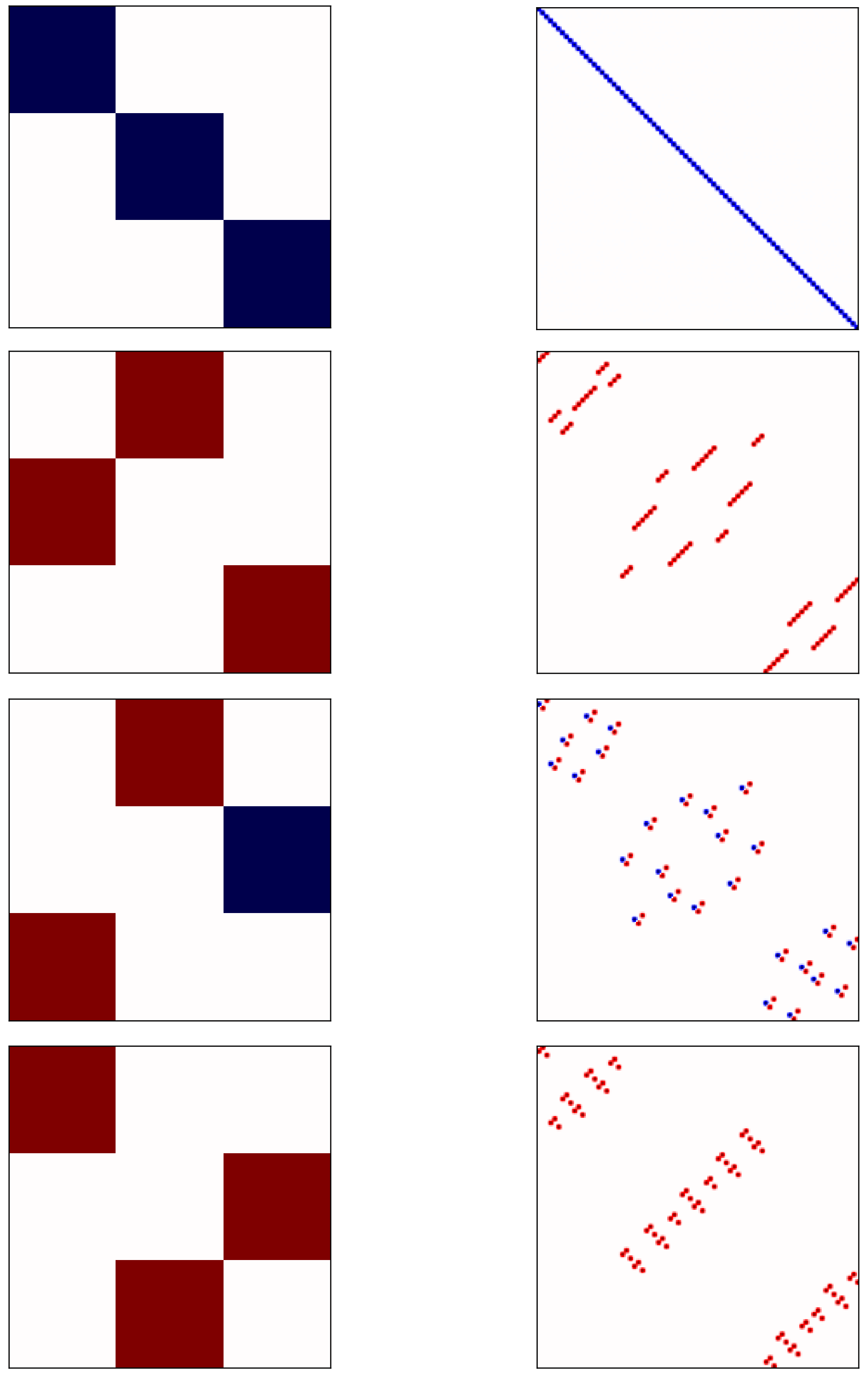
☐ Extract the induced representation of rotations & sum over irrep coefficients



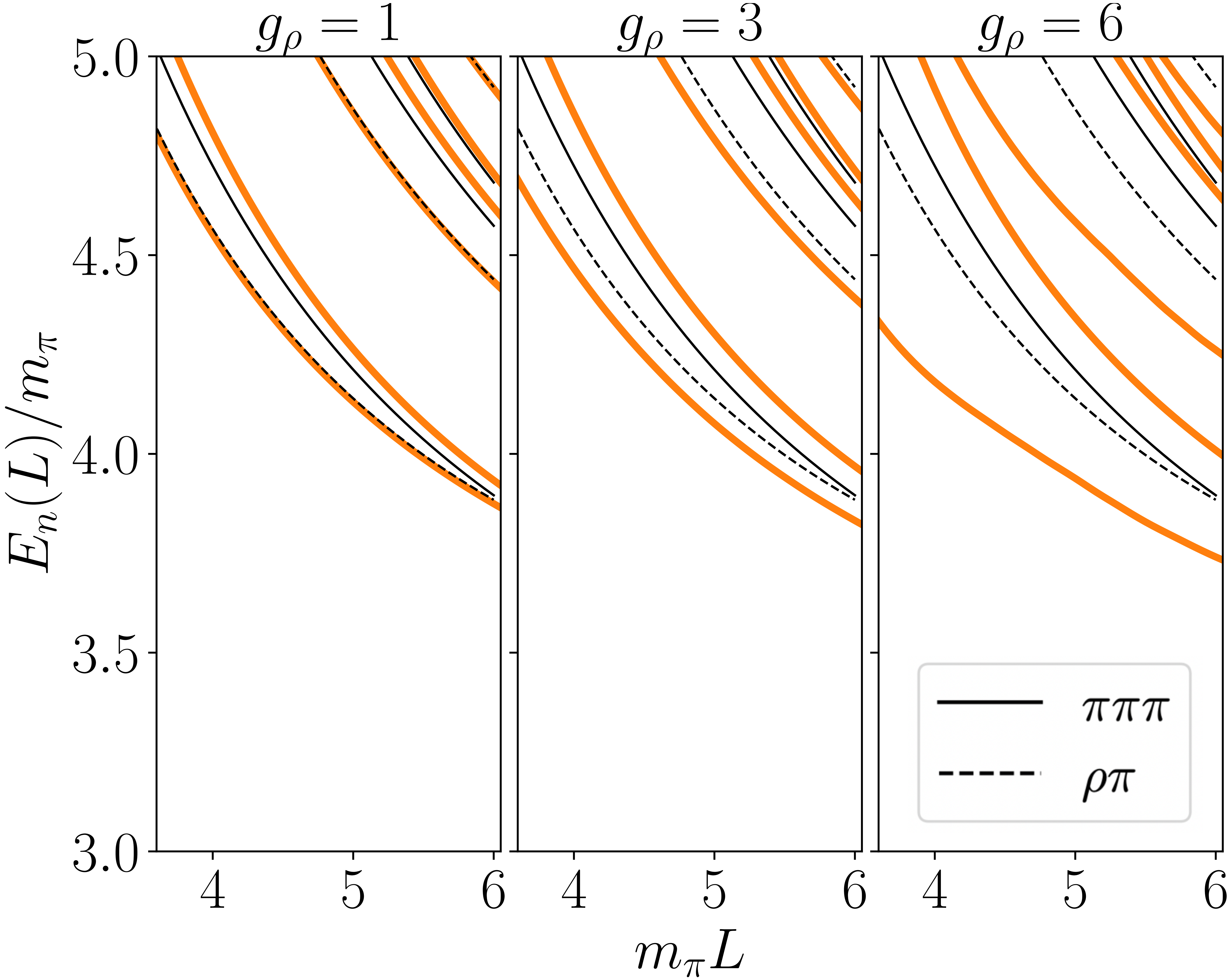
```
kellm space has size 108

    A1PLUS  covers 7x1 = 7 slots
    A2PLUS  covers 1x1 = 1 slots
    EPLUS   covers 6x2 = 12 slots
    T1PLUS  covers 4x3 = 12 slots
    T2PLUS  covers 7x3 = 21 slots
    A2MINUS covers 3x1 = 3 slots
    EMINUS  covers 2x2 = 4 slots
    T1MINUS covers 11x3 = 33 slots
    T2MINUS covers 5x3 = 15 slots

Total is 108
Total matches size of kellm space
```



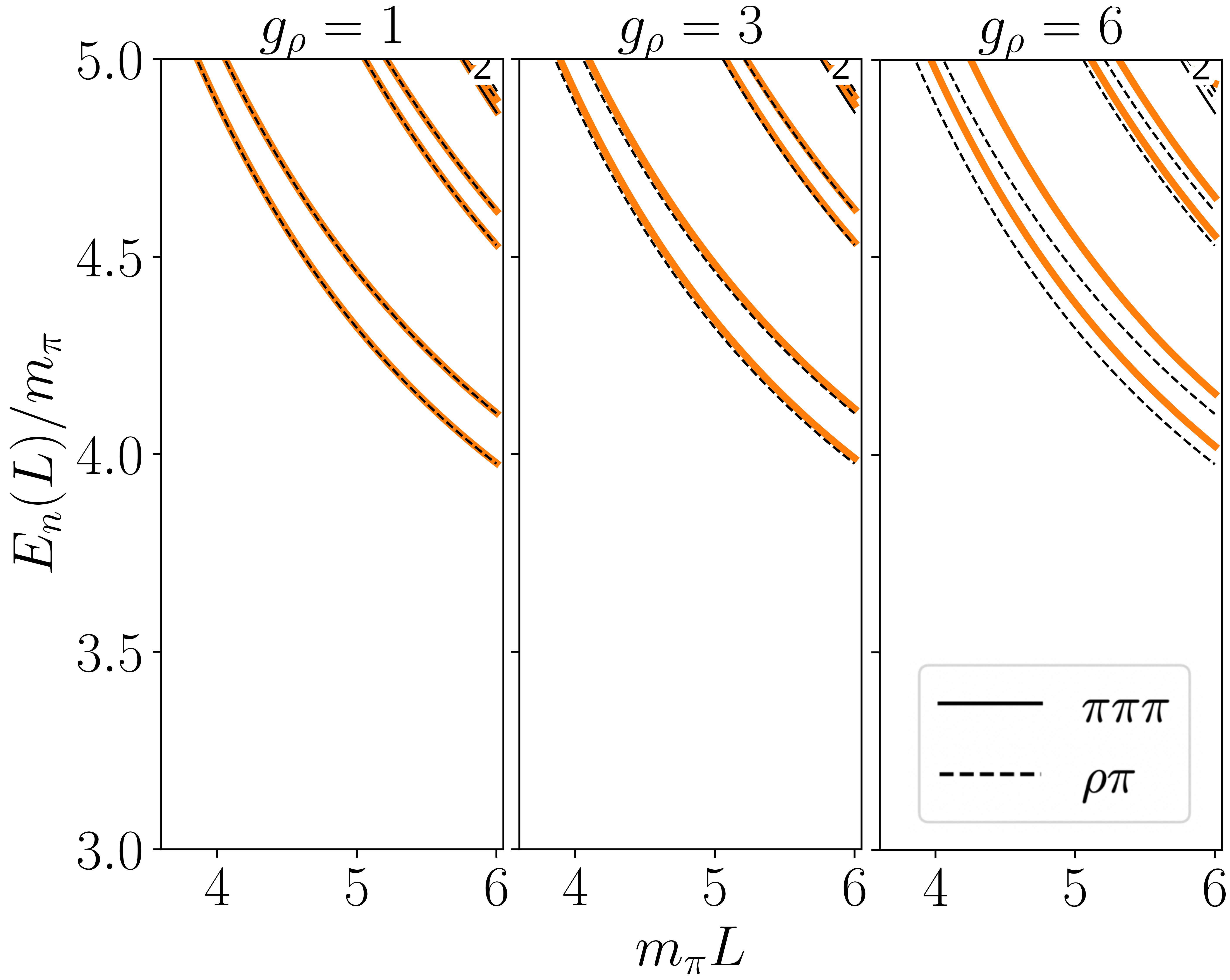
$I_{\pi\pi\pi} = 2, \;\; \boldsymbol{P} = [000], \;\; A_1^+$



- ☐ Result for $\mathbf{K}_{\mathrm{df},3} = 0$
- ☐ Truncate and parametrize:
  - — $I_{\pi\pi} = 2, \quad \ell = 0, \quad p \cot \delta_0(p) = -1/a_0$
  - — $I_{\pi\pi} = 1, \quad \ell = 1, \quad p \cot \delta_1(p) = \mathrm{BW}(m_\rho, g_\rho)$
- ☐ In this case: $m_\pi a_0 = 0.3, \;\; m_\rho/m_\pi = 2.2$ and g_ρ varies as shown

Alotaibi, Briceño, Hansen

$I_{\pi\pi\pi} = 2, \quad \boldsymbol{P} = [001], \quad A_1$



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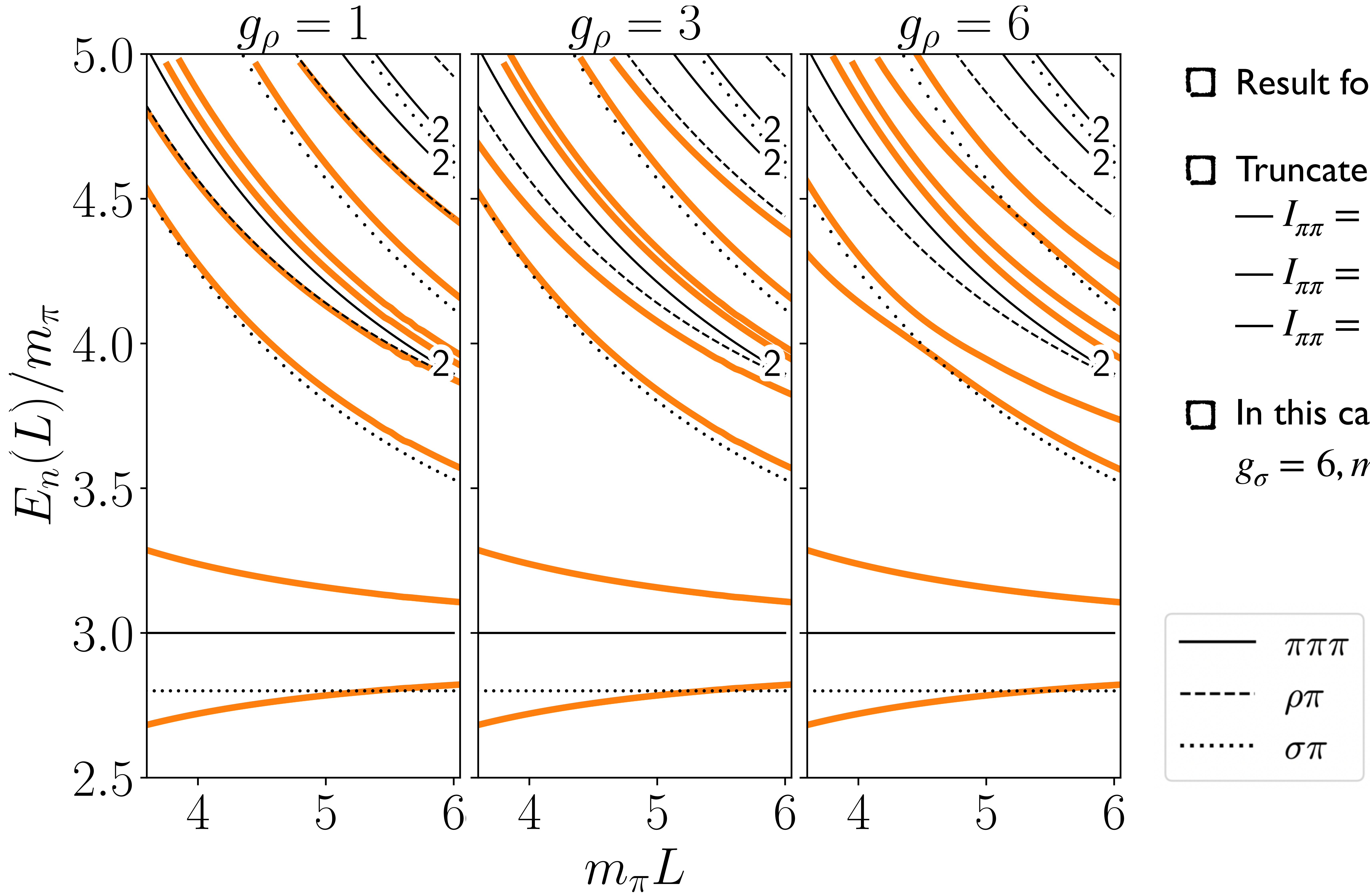
Alotaibi, Briceño, Hansen

$I_{\pi\pi\pi} = 1, \quad \boldsymbol{P} = [000], \quad A_1^-$



☐ Result for $\mathbf{K}_{\mathrm{df},3} = 0$

☐ Truncate and parametrize:

— $I_{\pi\pi} = 2, \quad \ell = 0, \quad p\cot\delta_0(p) = -1/a_0$

— $I_{\pi\pi} = 1, \quad \ell = 1, \quad p\cot\delta_1(p) = \mathrm{BW}(m_\rho, g_\rho)$

— $I_{\pi\pi} = 0, \quad \ell = 0, \quad p\cot\delta_1(p) = \mathrm{BW}(m_\sigma, g_\sigma)$

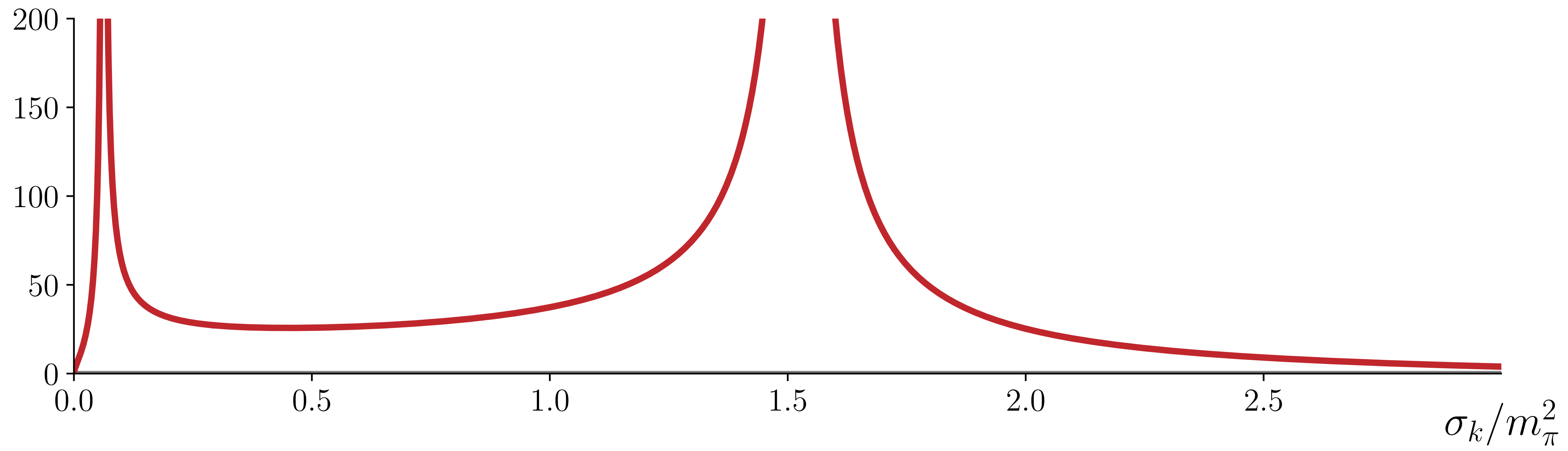
☐ In this case: $m_\pi a_0 = 0.3, \quad m_\rho/m_\pi = 2.2$

$g_\sigma = 6, m_\sigma/m_\pi = 1.8$

Alotaibi, Briceño, Hansen

A few technical comments

☐ Breit-Wigner can have unphysical, deep subthreshold poles → must be removed by our smooth cutoff



Note: also physical pole requires care but is understood

☐ We have lots of freedom in our choice for $\mathscr{K}_{\mathrm{df},3}$…. including

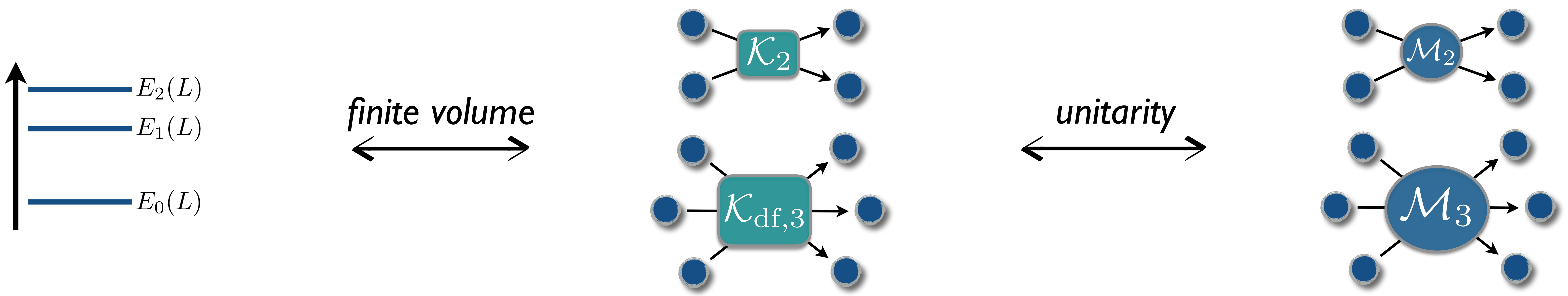
the exact cutoff functions

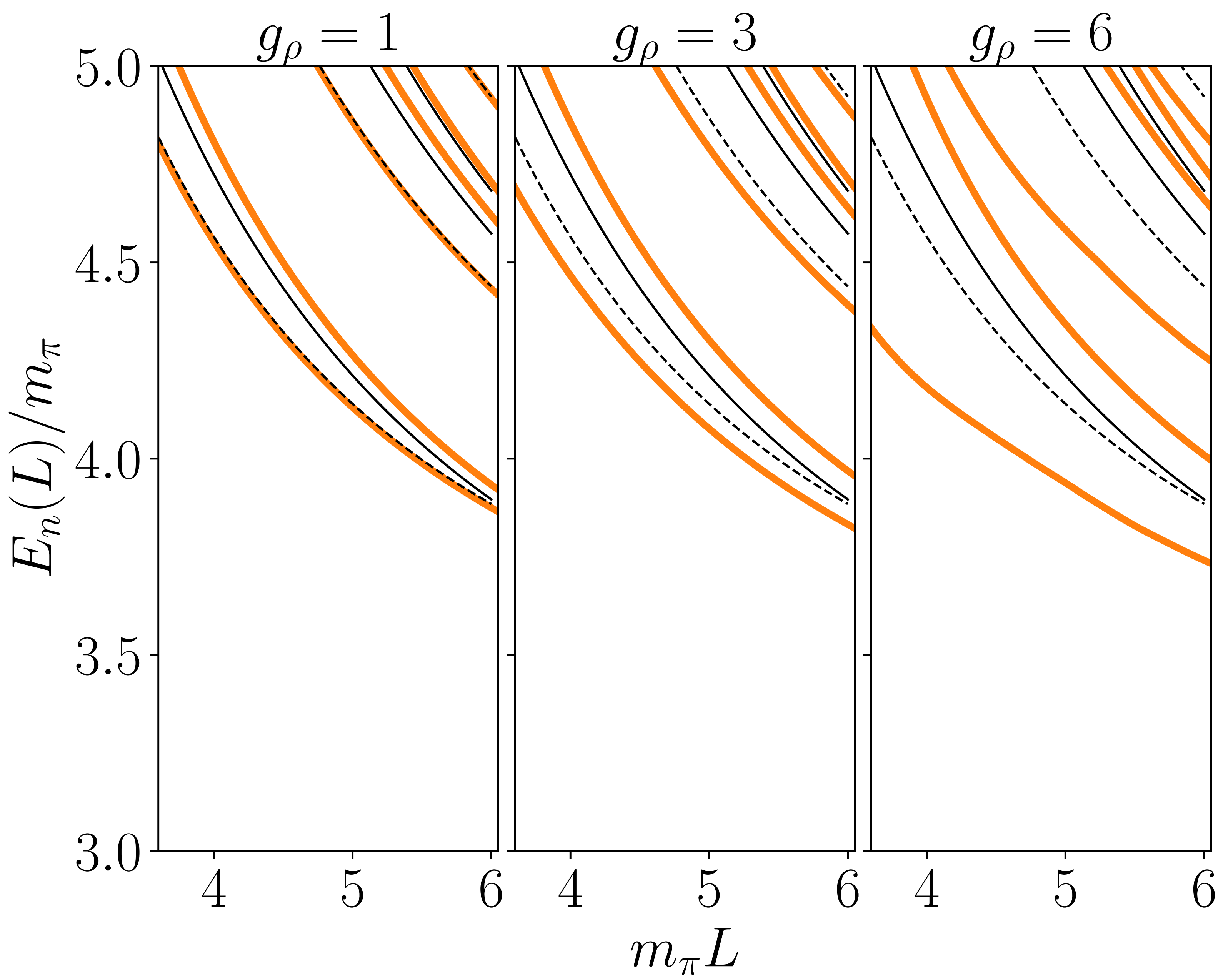
the exact behaviour of $\mathscr{K}_2$ below threshold

the removal of physical poles

symmetric vs asymmetric

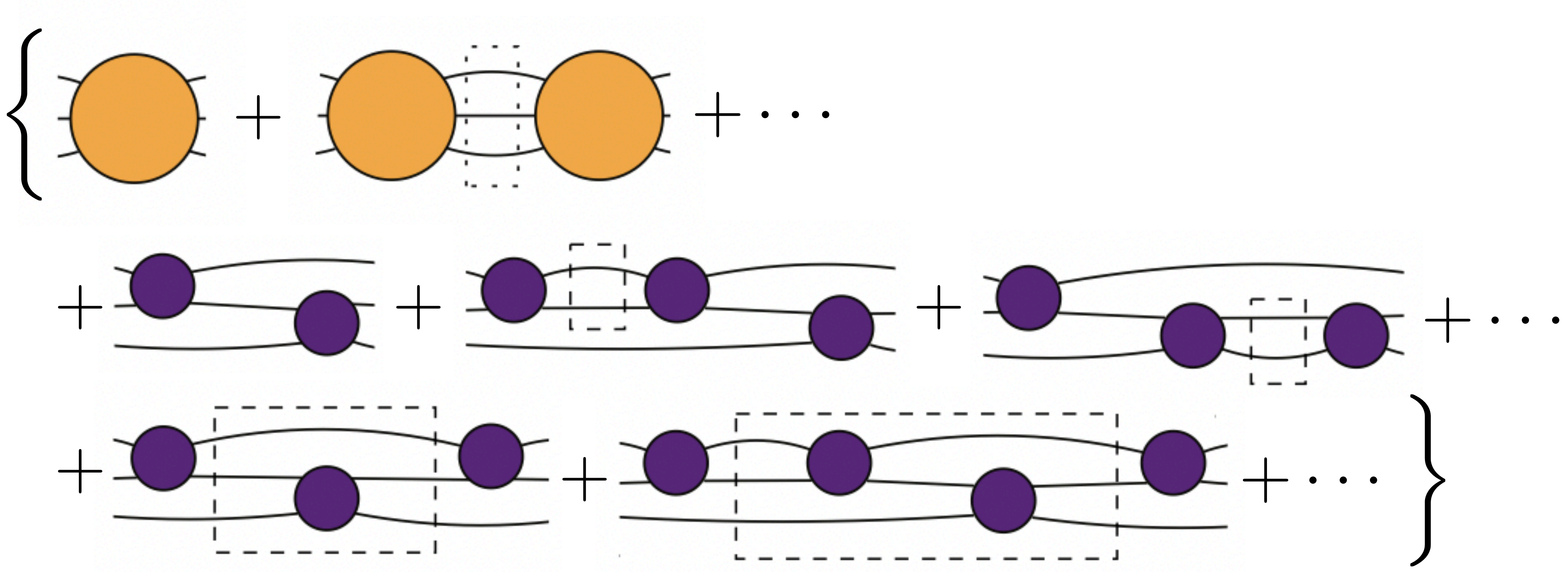
From the finite-volume energies to amplitudes





From K-matrices to amplitudes

$$\mathcal{M}_3^{\mathsf{un}} = \lim_{L\to\infty}$$

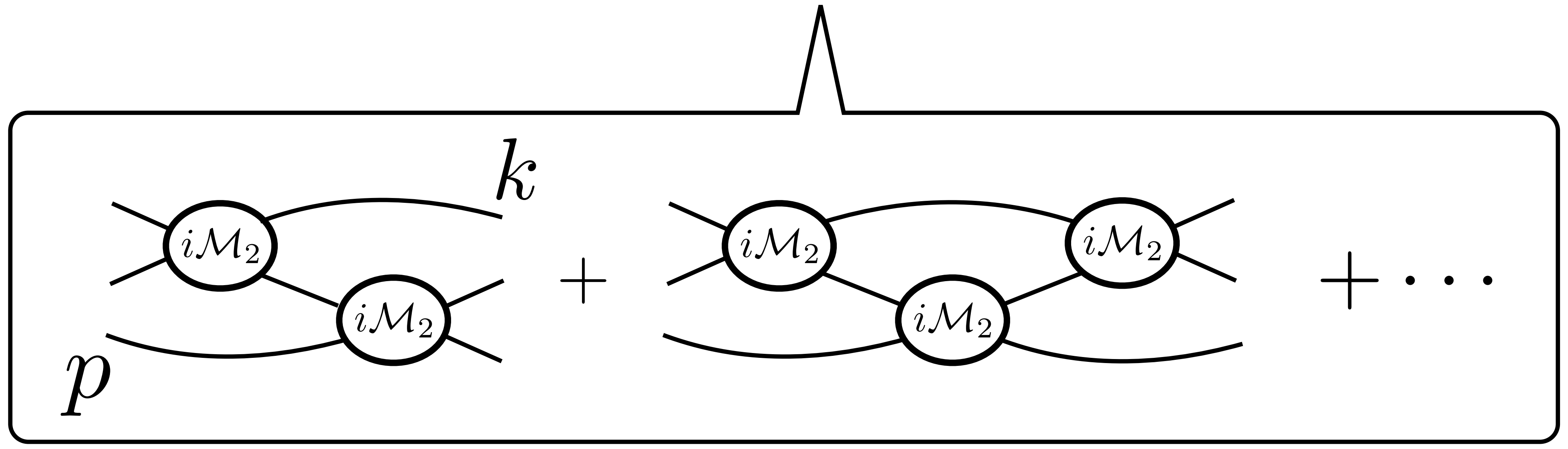


- ☐ Imitate much of the finite-volume decomposition with alternative "endcaps"
- ☐ Recover an analytic expression for which we can send $L \to \infty$

$$\mathcal{M}_3^{\mathsf{un}} = \mathcal{I}[\mathcal{K}_2, \mathcal{K}_{\mathsf{df,3}}]$$

the scattering amplitude is a known functional of the on-shell K-matrices

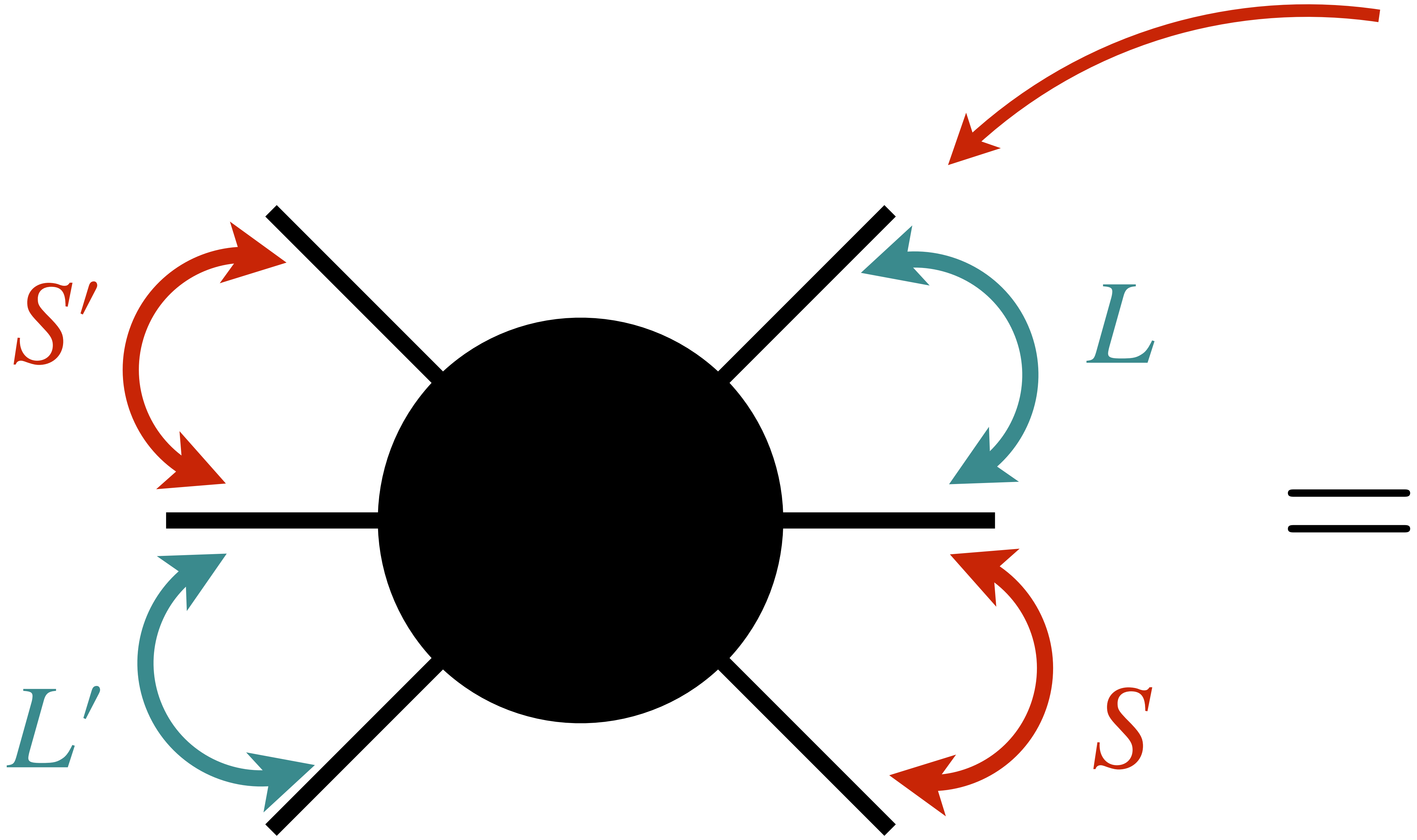
$$\mathcal{M}_3^{\mathsf{un}}(E_3^\star, \boldsymbol{p}, \boldsymbol{k}) = \mathcal{D}^{\mathsf{un}}(E_3^\star, \boldsymbol{p}, \boldsymbol{k}) + \mathcal{E}^{\mathsf{un}}(E_3^\star, \boldsymbol{p})\mathcal{T}(E_3^\star)\mathcal{E}^{\mathsf{un}}(E_3^\star, \boldsymbol{k})$$



Vanishes for $K_{\text{df},3} = 0$

MTH, Sharpe (2014-2016)

Angular momentum projection



this requires a choice in an “spectator”

$$= i \left[\mathcal{M}_3^{J^P} \right]_{L'S',LS}$$

Briceño

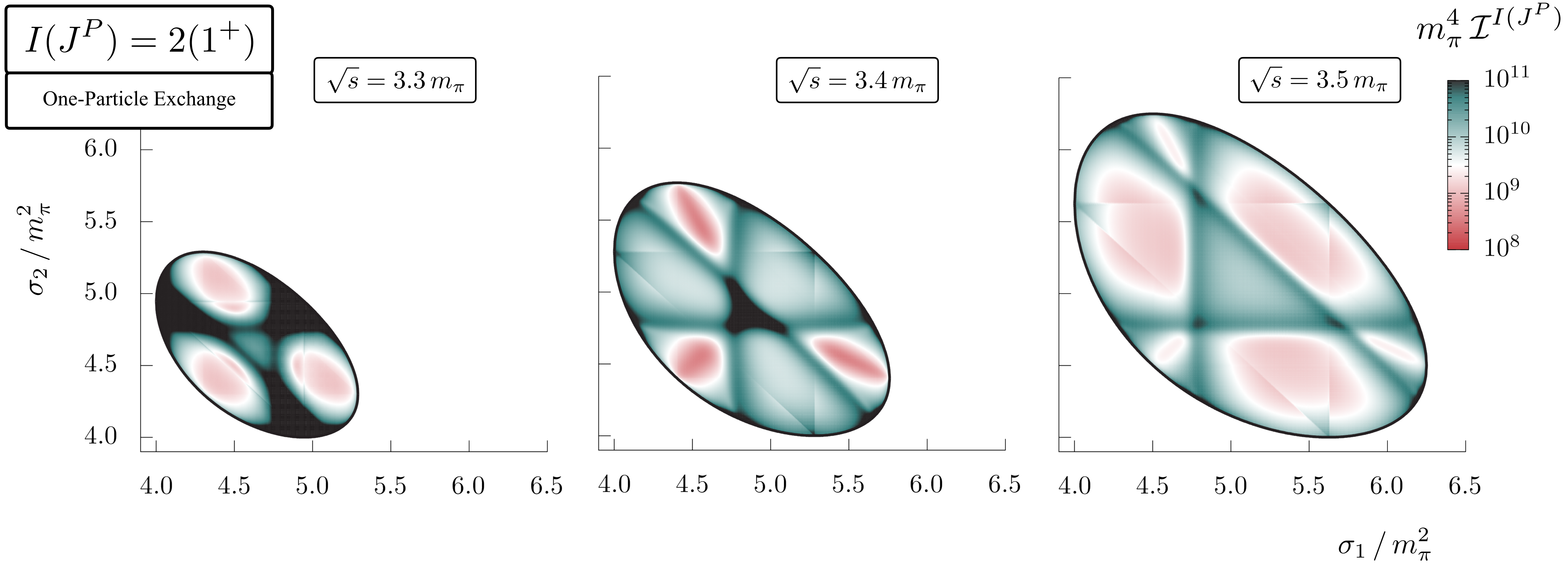
S. R. Costa

Jackura

Costa, Jackura, Briceño (2024)

Symmetrization through recoupling coefficients

$$\mathcal{M}^{\,J^P} = \sum_{j',j} \mathcal{R}_{j'}\, \mathcal{M}^{(j',j)\;J^P}\, \mathcal{R}_j$$



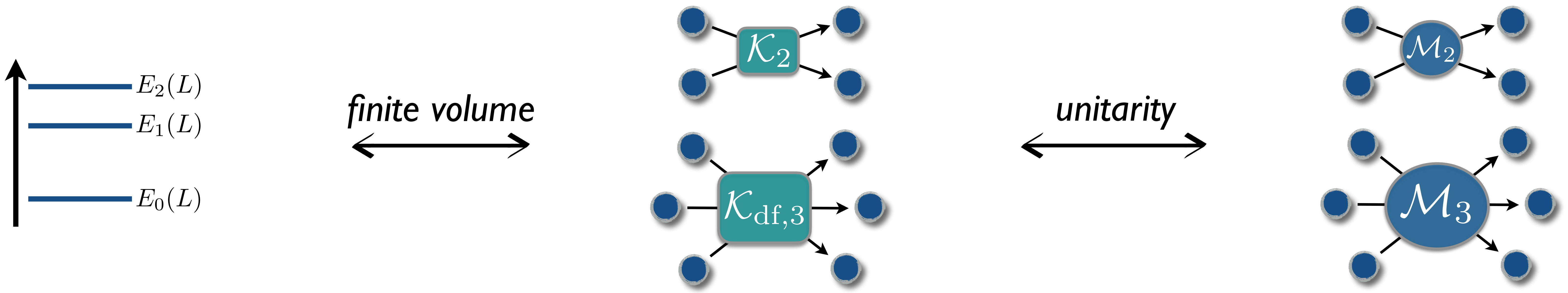
Briceño

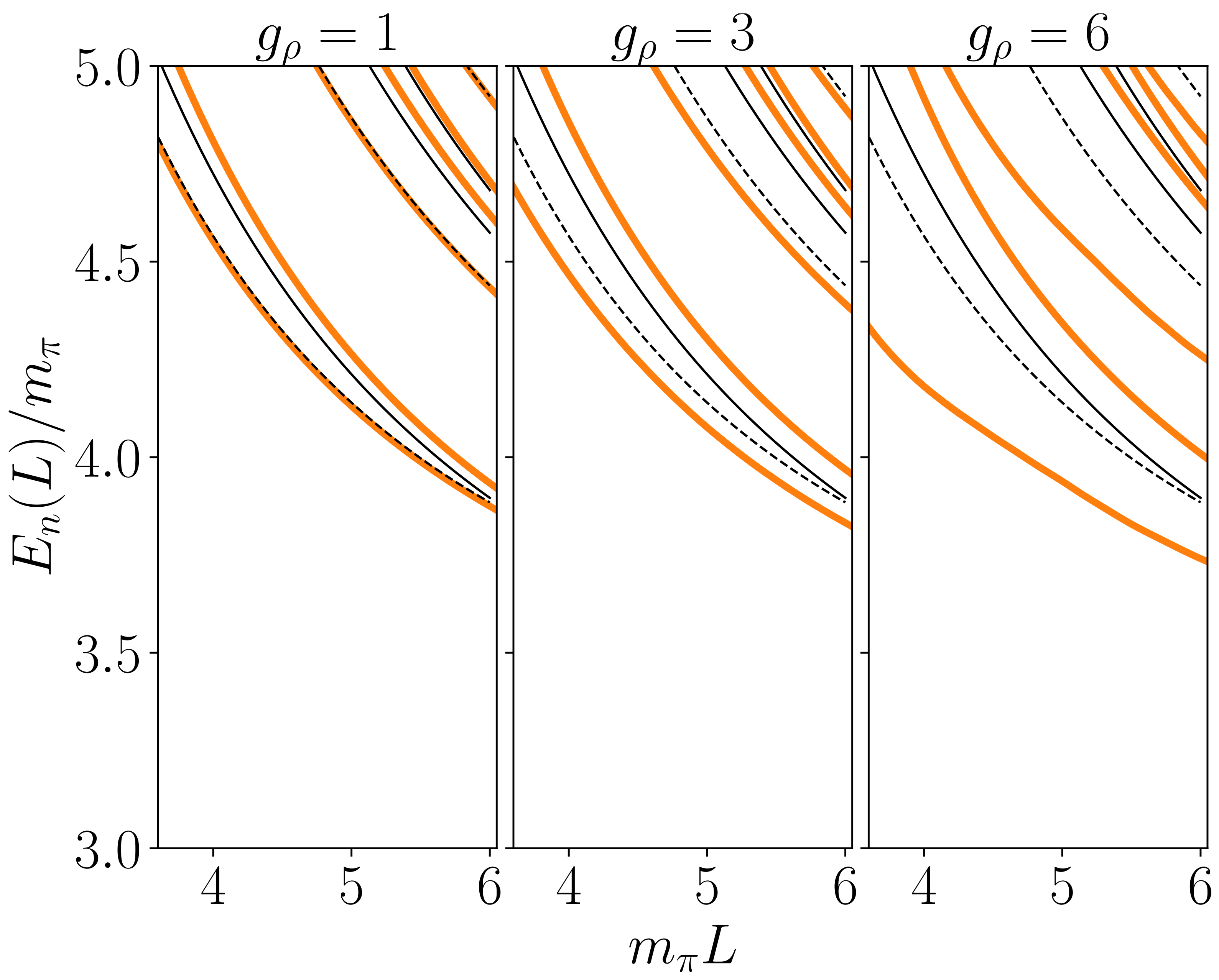
Chambers

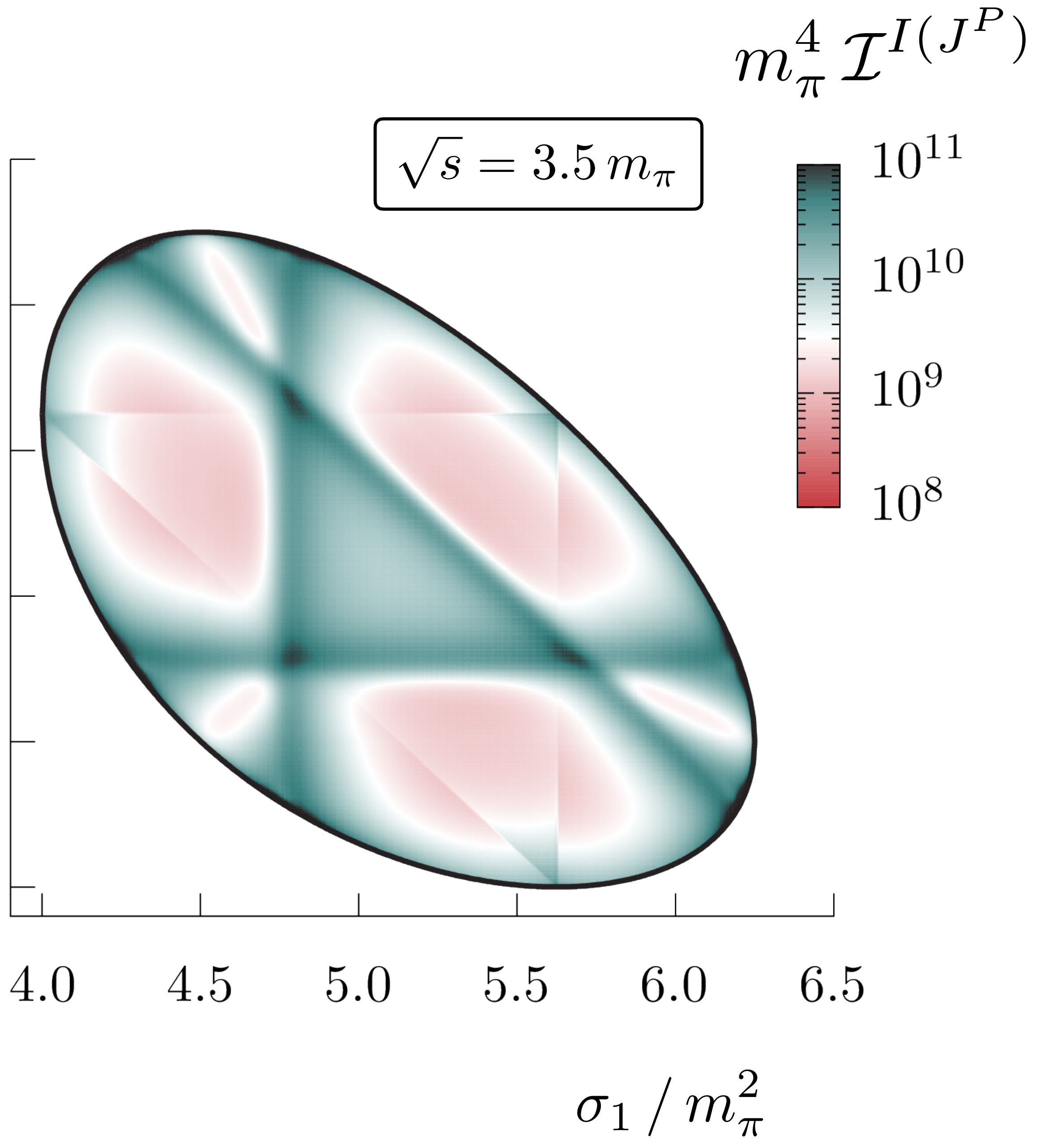
Jackura

Briceño, Chamber,s Jackura (2025)… see also Dawid, Romero-Lopez, Sharpe

From the finite-volume energies to amplitudes



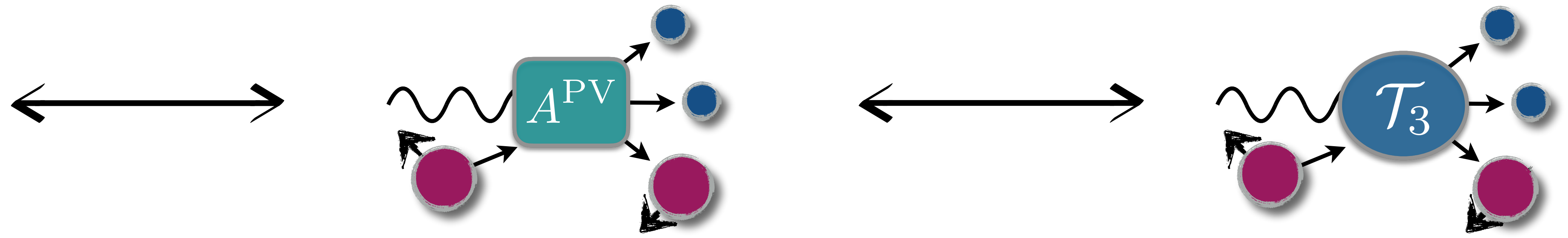




Summary and outlook

- ☐ RFT formalism already describes many interesting systems: $(\pi\pi\pi)_{I_{\pi\pi\pi}}$, $(\pi\pi K)$, $(\pi K K)$
- ☐ Formalism for general three-particle states is "around the corner" (also for decays)

$\langle E_n, L|\mathcal{J}|N\rangle$



- ☐ Analysis is getting expensive!
- ☐ Strategies to speed things up:

  prepare quantization condition space as much as possible!

  group theory → prepare projectors (various tricks when projection is planned)

- ☐ Major recent progress on integral equations

Thanks for listening!