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PRD 108 034513 (2023)
PRD 109 034513 (2024)

$\pi\pi$ scattering & the σ from lattice QCD – dispersing some confusion ?

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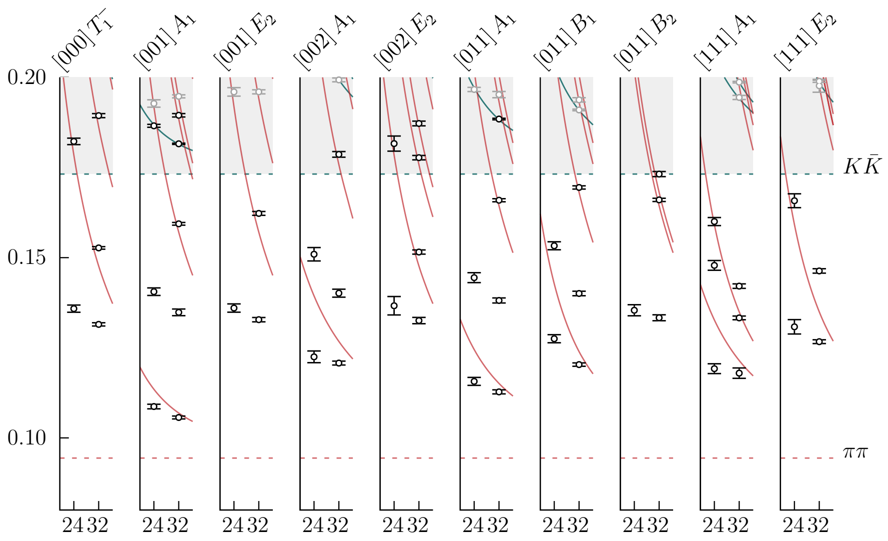
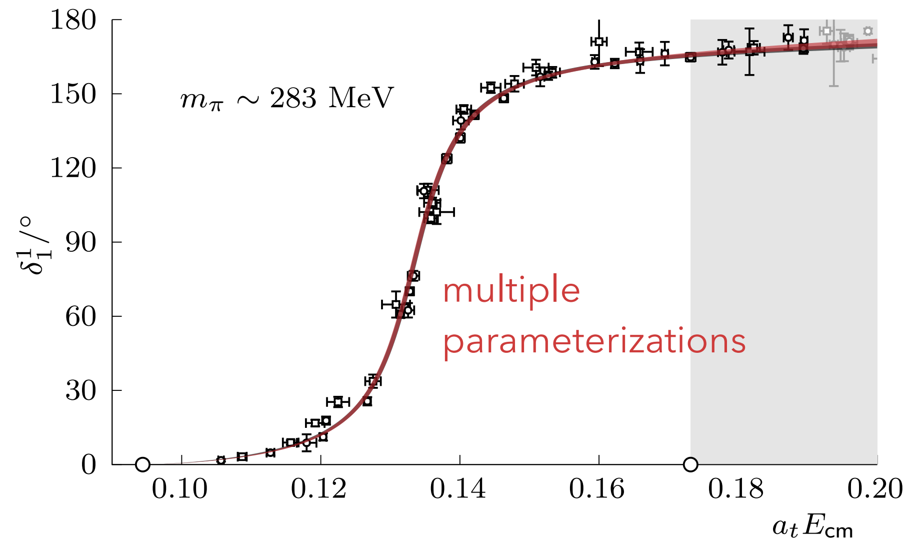
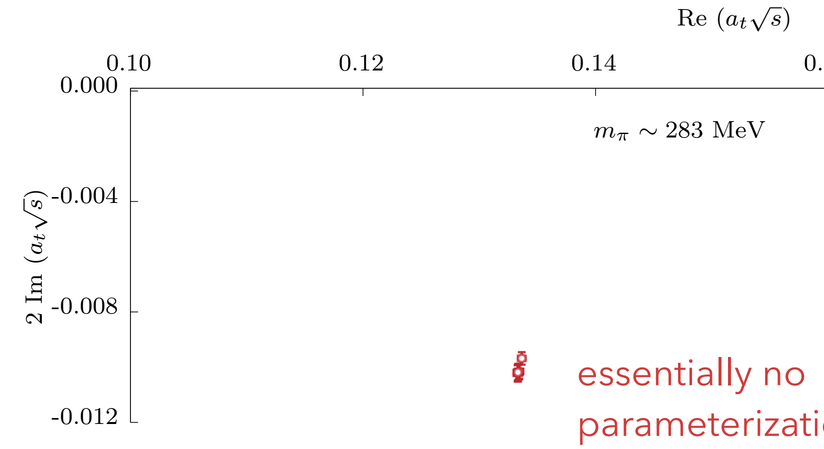
$\pi\pi$ elastic scattering in P-wave

PRD 108, 034513 (2023)







essentially no
parameterization
dependence

expt.

$\pi\pi$ elastic scattering in P-wave

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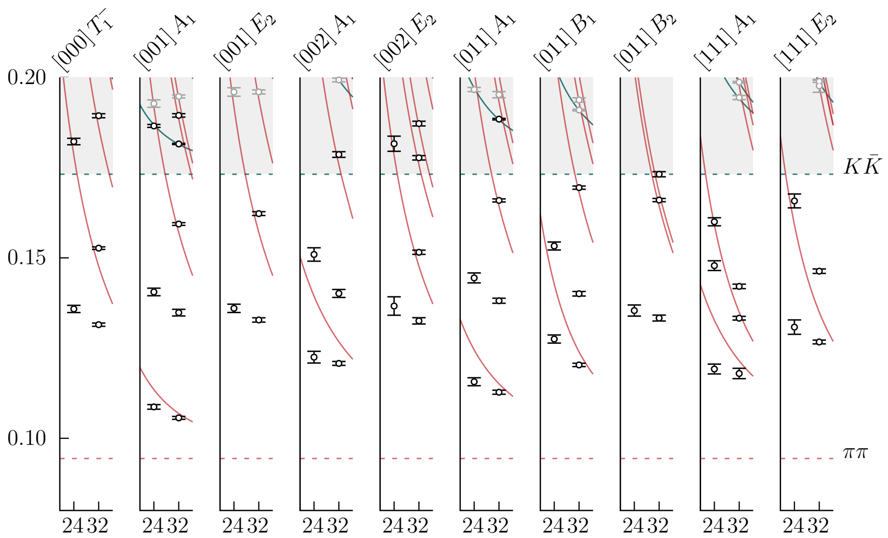
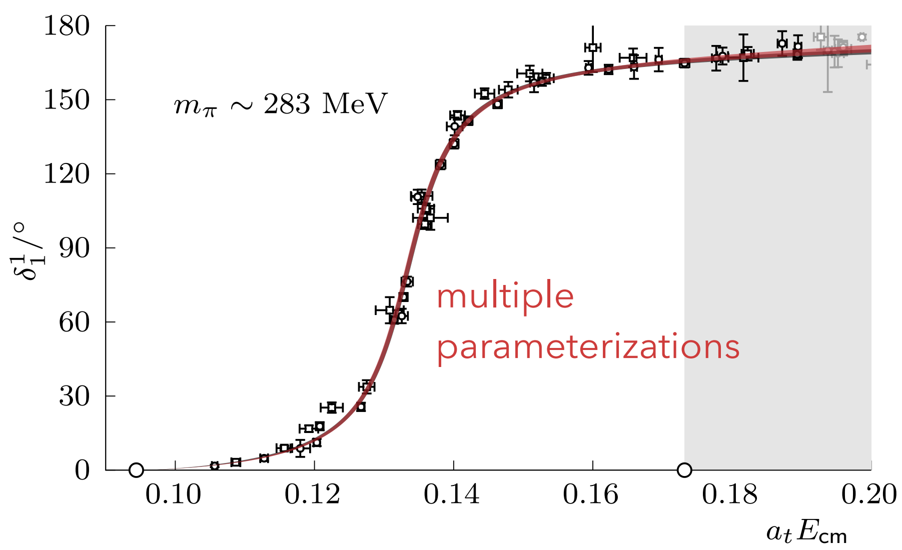
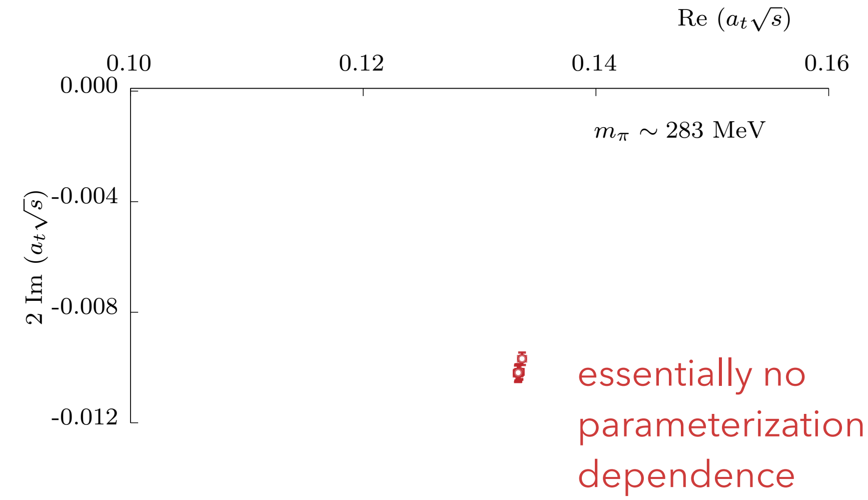
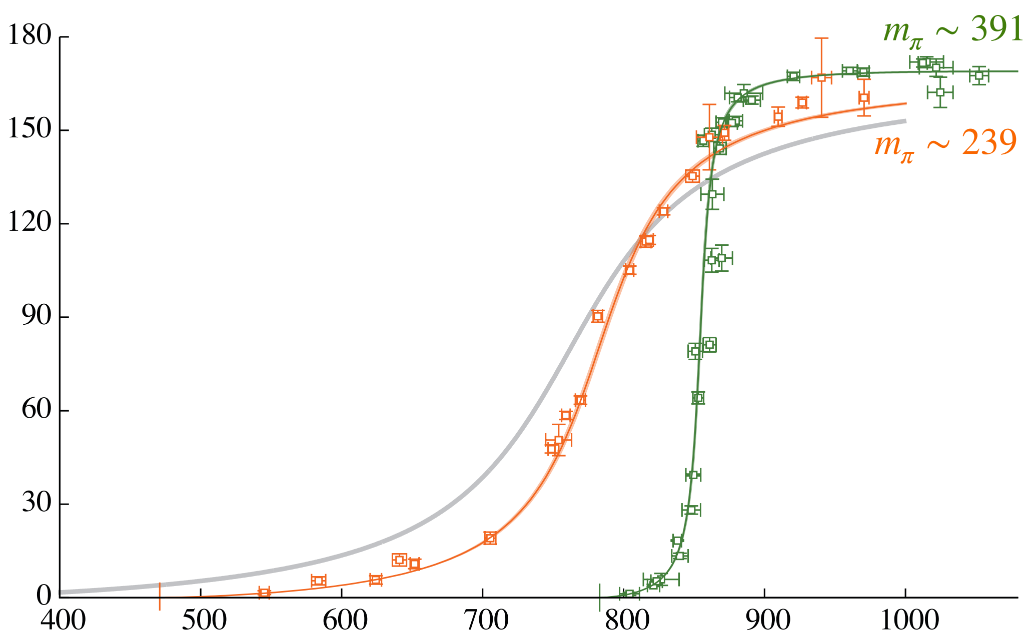
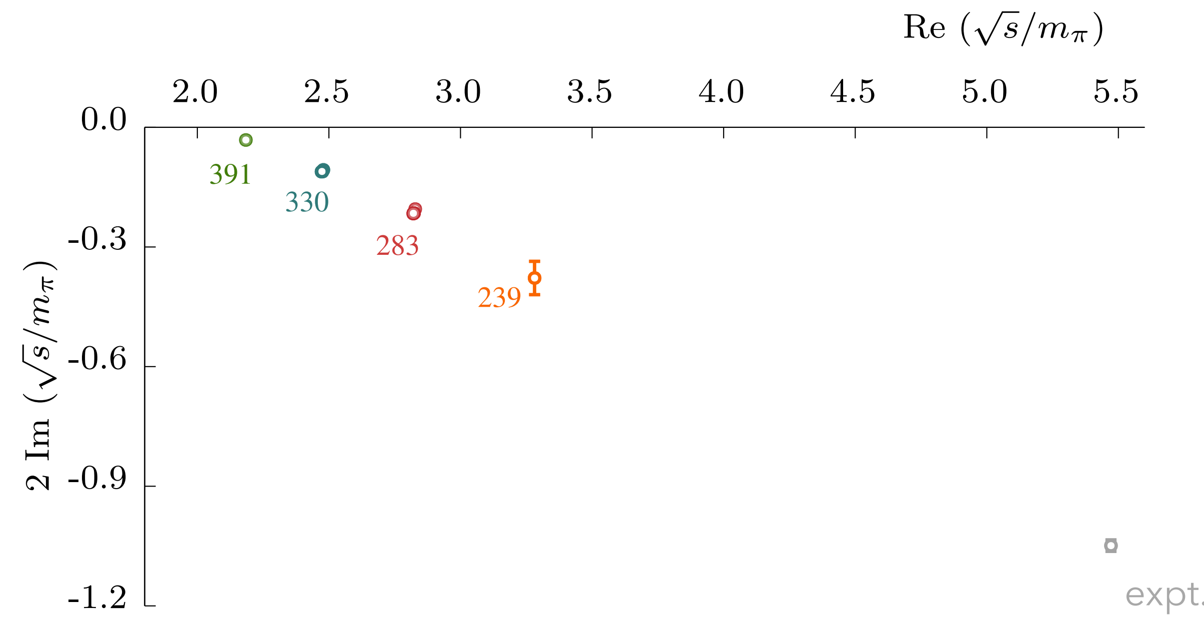




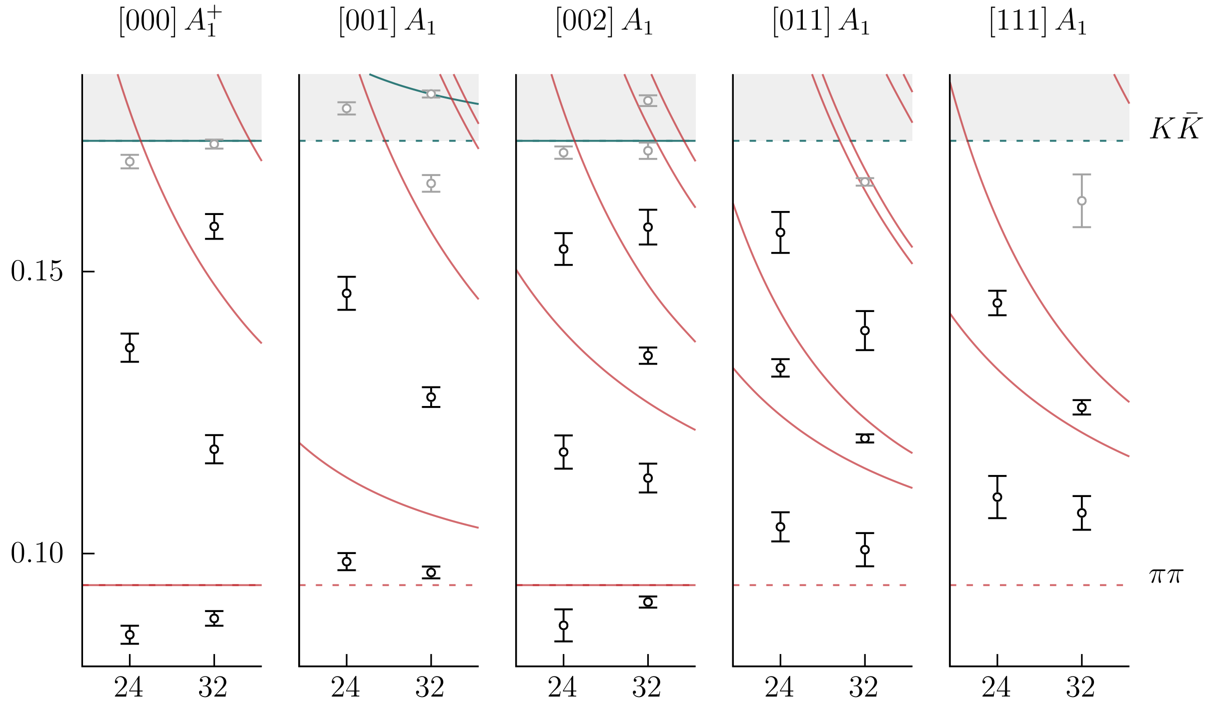


same story at several pion masses – simple quark mass evolution





$\pi\pi$ elastic scattering in S-wave



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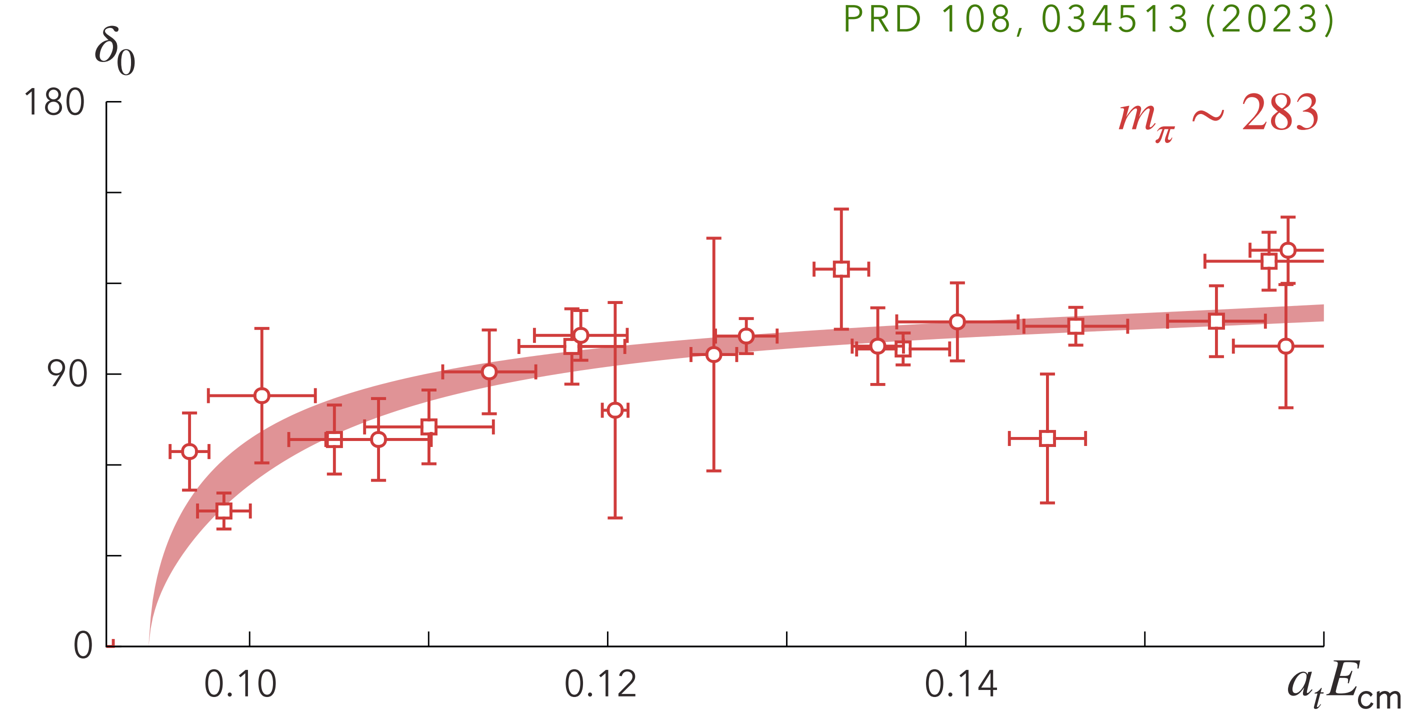
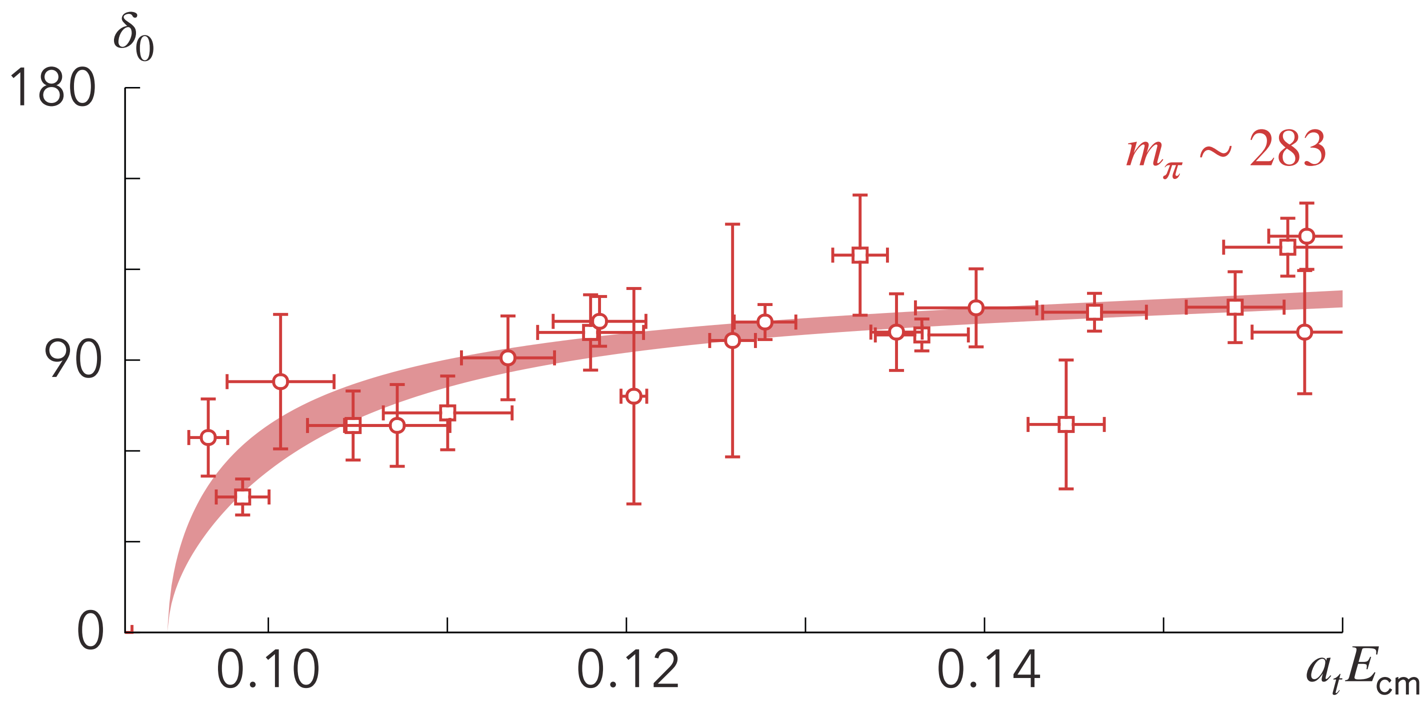
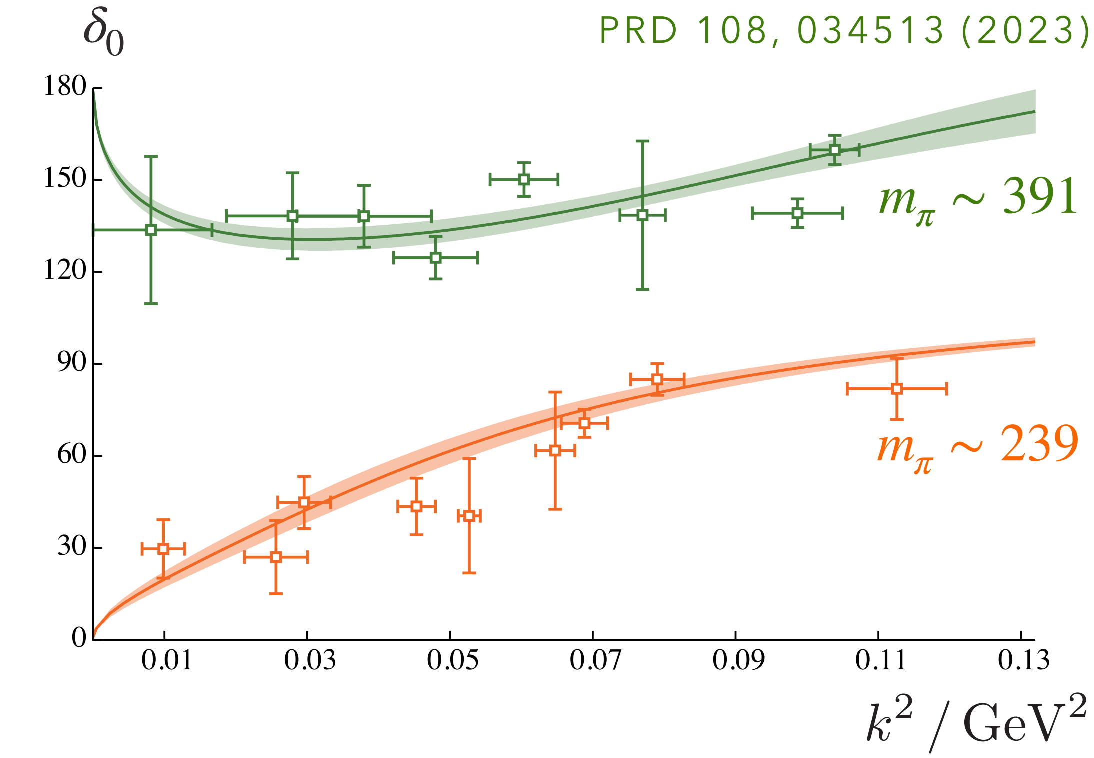
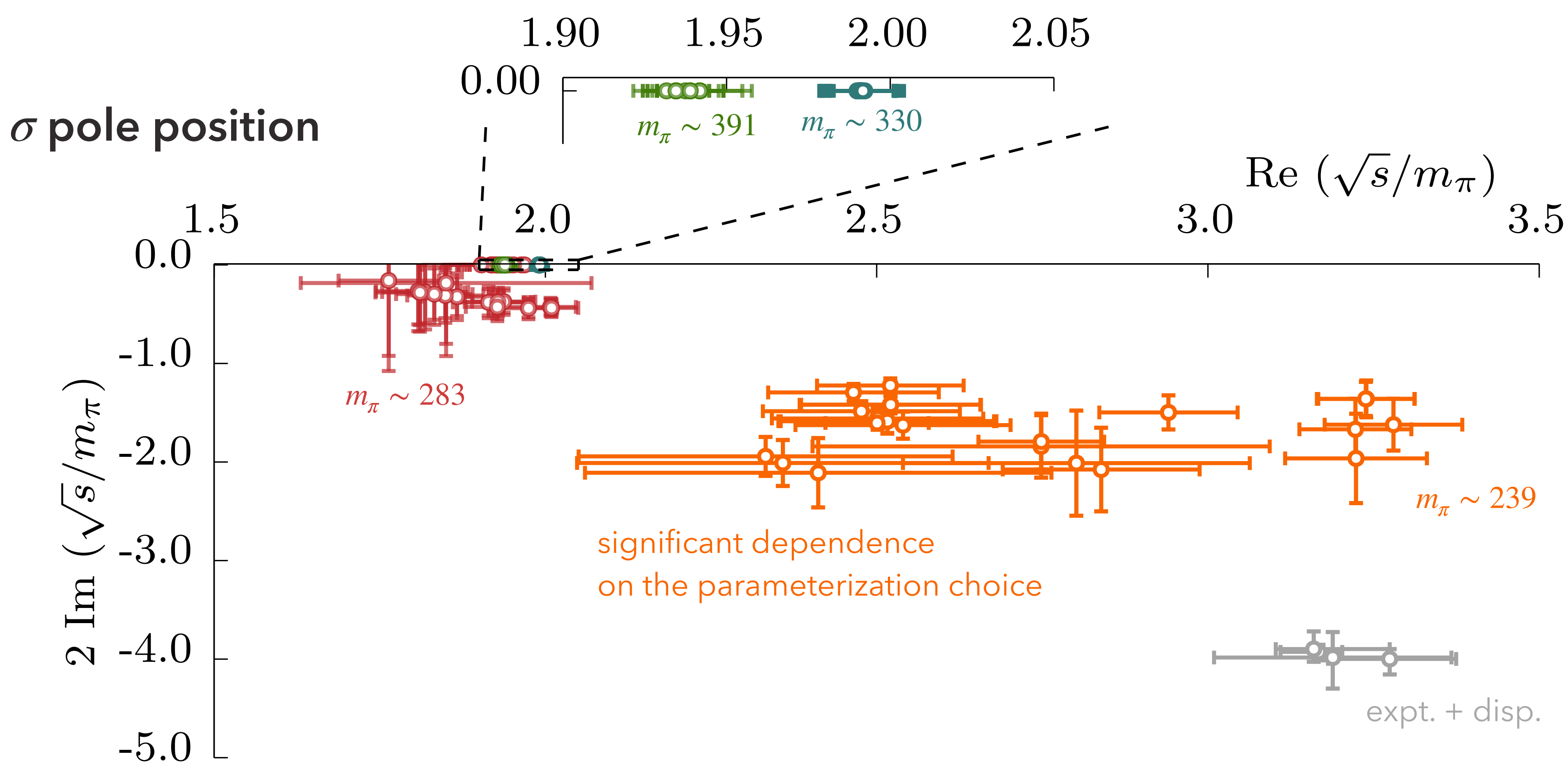


$\pi\pi$ elastic scattering in S-wave







other lattice $\pi\pi$ scattering determinations are available, only *hadspec* data used here

what are these amplitude parameterizations you're using ?

in **coupled-channel cases**, generally ***K***-matrix forms – ensure **exact unitarity**

required by f.v. quantization condition

$$[t^{-1}]_{ij}(s) = [K^{-1}]_{ij}(s) - i\rho_i(s)\delta_{ij}$$

phase space $\rho_i(s) = \dfrac{2k_i(s)}{\sqrt{s}}$

K-matrix real and symmetric, commonly poles + polynomials

$$K_{ij}(s) = \sum_p \frac{c_i^{(p)}\, c_j^{(p)}}{s_p - s} + \sum_n \gamma_{ij}^{(n)} s^n$$

very flexible parameterization

or slightly improving the analytic properties, a dispersively improved Chew-Mandelstam phase-space

$$[t^{-1}]_{ij}(s) = [K^{-1}]_{ij}(s) + I_i(s)\delta_{ij}$$

$$I_i(s) = I_i(s_0) + \frac{s - s_0}{\pi} \int_{s_{\mathrm{thr}}}^{\infty} ds' \frac{-\rho_i(s')}{(s' - s_0)(s' - s)}$$

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in **elastic case**, easy to use wider variety …

(variants) of effective range expansion: $k^{2\ell+1}\cot\delta_\ell = F_\ell(s)\left(\frac{1}{\hat{a}_\ell} + \frac{1}{2}\hat{r}_\ell k^2 + \ldots\right)$

can enforce subthreshold zeroes with $F = \frac{4m_\pi^2 - s_A}{s - s_A}$

a conformal mapping in energy: $k^{2\ell+1}\cot\delta_\ell = \frac{\sqrt{s}}{2} F_\ell(s) \sum_n B_n \omega^n$

$$\omega(s) \equiv \frac{\sqrt{s} - \alpha\sqrt{s_0 - s}}{\sqrt{s} + \alpha\sqrt{s_0 - s}}$$

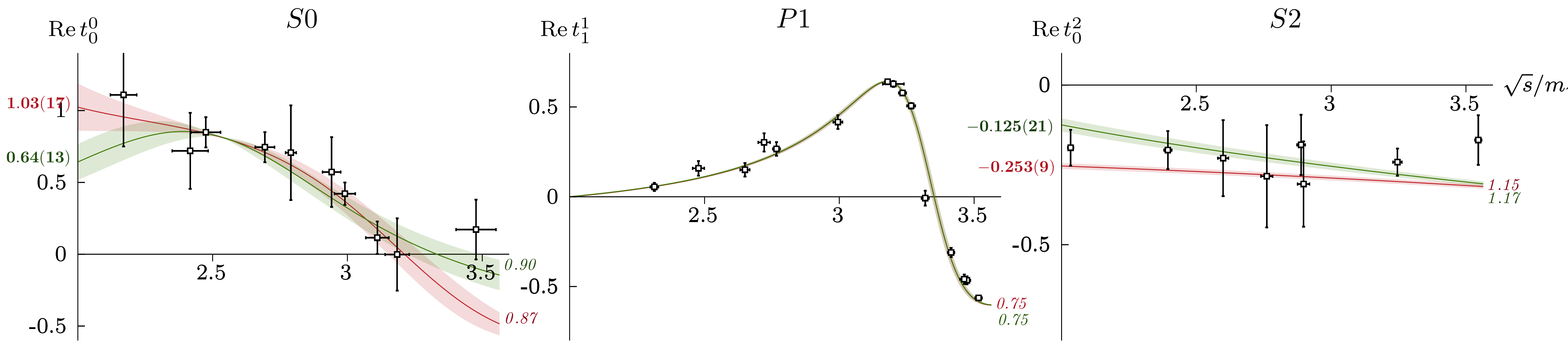
also Breit-Wigner, ***K***-matrix as above …

philosophy: only trying to directly parameterize along the real energy axis above threshold
avoiding inserting any prior beliefs beyond unitarity – let the data tell us what needs to be there

amplitude parameterization variation at $m_\pi \sim 239\,\mathrm{MeV}$

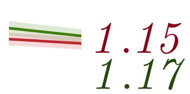
$I = 0,\ \ell = 0$
two example amplitude descriptions

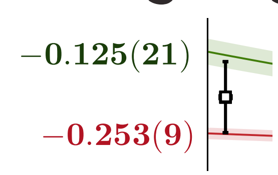
$I = 2,\ \ell = 0$
two example amplitude descriptions



in each S-wave case,
comparable χ^2/N_{dof} but rather different **scattering lengths**

(σ pole location also differs)



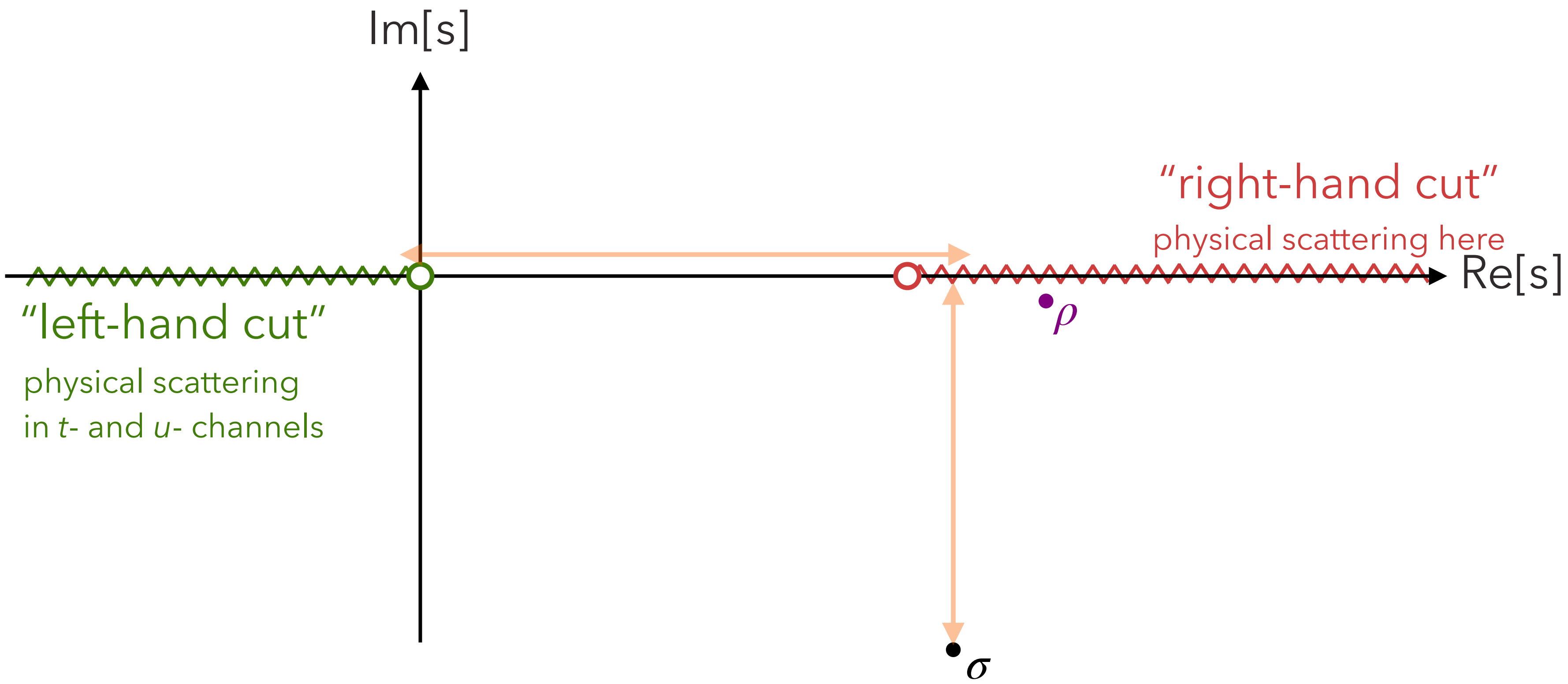


what's going on?

why are the scalar partial-waves subject to so much parameterization variation?

are we missing some important feature of the amplitudes?

scattering amplitude in the complex energy plane

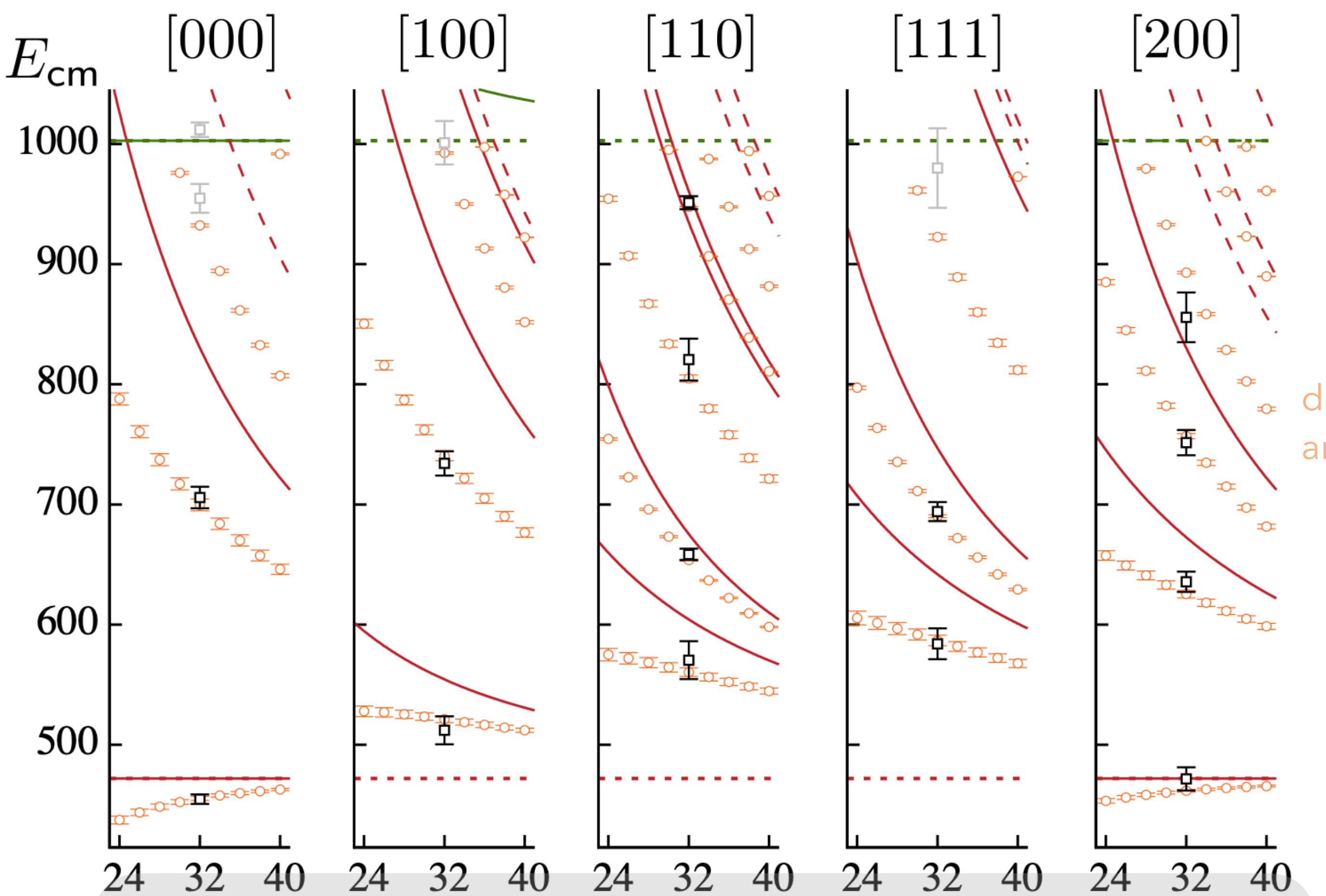


for **broad** resonances, the left-hand cut may be just as close to physical scattering – shouldn't neglect its influence

our simple K-matrix forms **don't have a left-hand cut**

how do we capture that physics in our amplitudes … ?

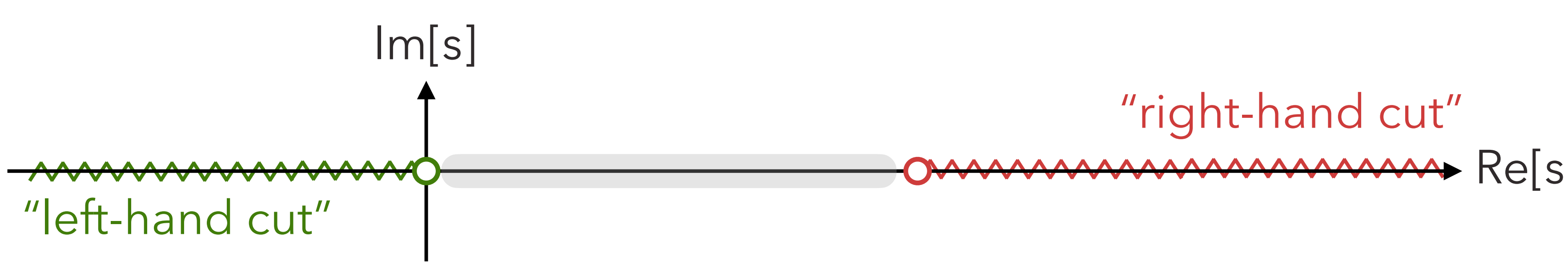
$\pi\pi$ $I = 0$ energy levels at $m_\pi \sim 239$ MeV



description with one of many amplitude parameterizations

no energy levels in this region,
no real constraint on subthreshold amplitude
or the left-hand cut

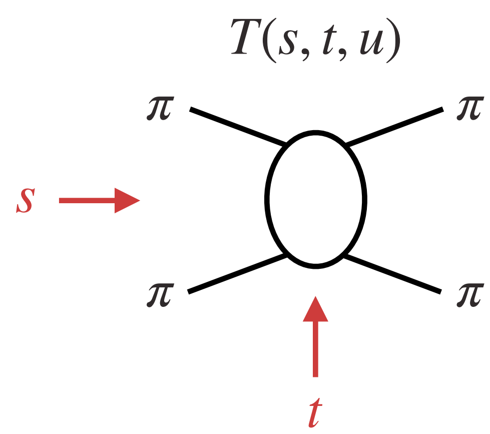
"left-hand cut"



can we construct amplitudes such that the left-hand cut is present and constrained?

we're treating different partial-waves as independent, should we ?

the crossed-channels of elastic $\pi\pi$ scattering are also elastic $\pi\pi$ scattering



partial-wave projected into s-channel

$$t_\ell(s) = \frac{1}{64\pi}\int_{-1}^{1} d\cos\theta_s \, T(s,t,u)\, P_\ell(\cos\theta_s)$$

crossing symmetry gets obscured by partial-wave projection

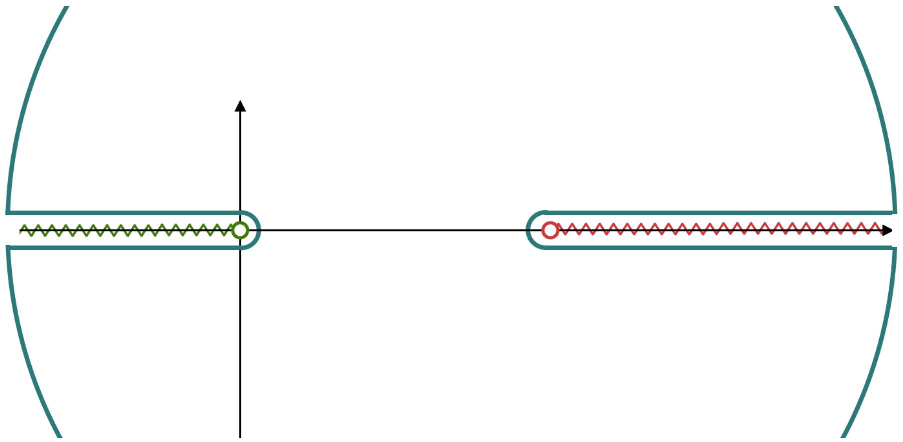
how to practically make use of the crossing symmetry ?

… use **analyticity** to form **dispersion relations**

… relates different partial-waves (in different isospins)

(longstanding approach using experimental phase-shift data, a.k.a "Roy equations")

dispersive implementation of crossing



"twice subtracted" dispersion relations

$$\tilde{t}^I_\ell(s) = \underline{\tau^I_\ell(s)} + \sum_{I',\ell'} \int_{s_{\text{thr.}}}^{\infty} ds'\, K^{II'}_{\ell\ell'}(s',s)\, \text{Im}\, t^{I'}_{\ell'}(s')$$

lattice QCD input: partial-wave amplitudes in $I = 0,1,2$

$$\tau^0_0(s)/m_\pi = \tfrac{1}{3}\big(a^0_0 + 5a^2_0\big) + \tfrac{1}{3}\big(2a^0_0 - 5a^2_0\big)\frac{s}{4m^2_\pi}$$

$$\tau^1_1(s)/m_\pi = \tfrac{1}{18}\big(2a^0_0 - 5a^2_0\big)\frac{s - 4m^2_\pi}{4m^2_\pi}$$

$$\tau^2_0(s)/m_\pi = \tfrac{1}{6}\big(2a^0_0 + a^2_0\big) - \tfrac{1}{6}\big(2a^0_0 - 5a^2_0\big)\frac{s}{4m^2_\pi}$$

low-order polynomials in terms of S-wave scattering lengths $a^I_0 \equiv \text{Re}\, t^I_0(s = 4m^2_\pi)/m_\pi$,

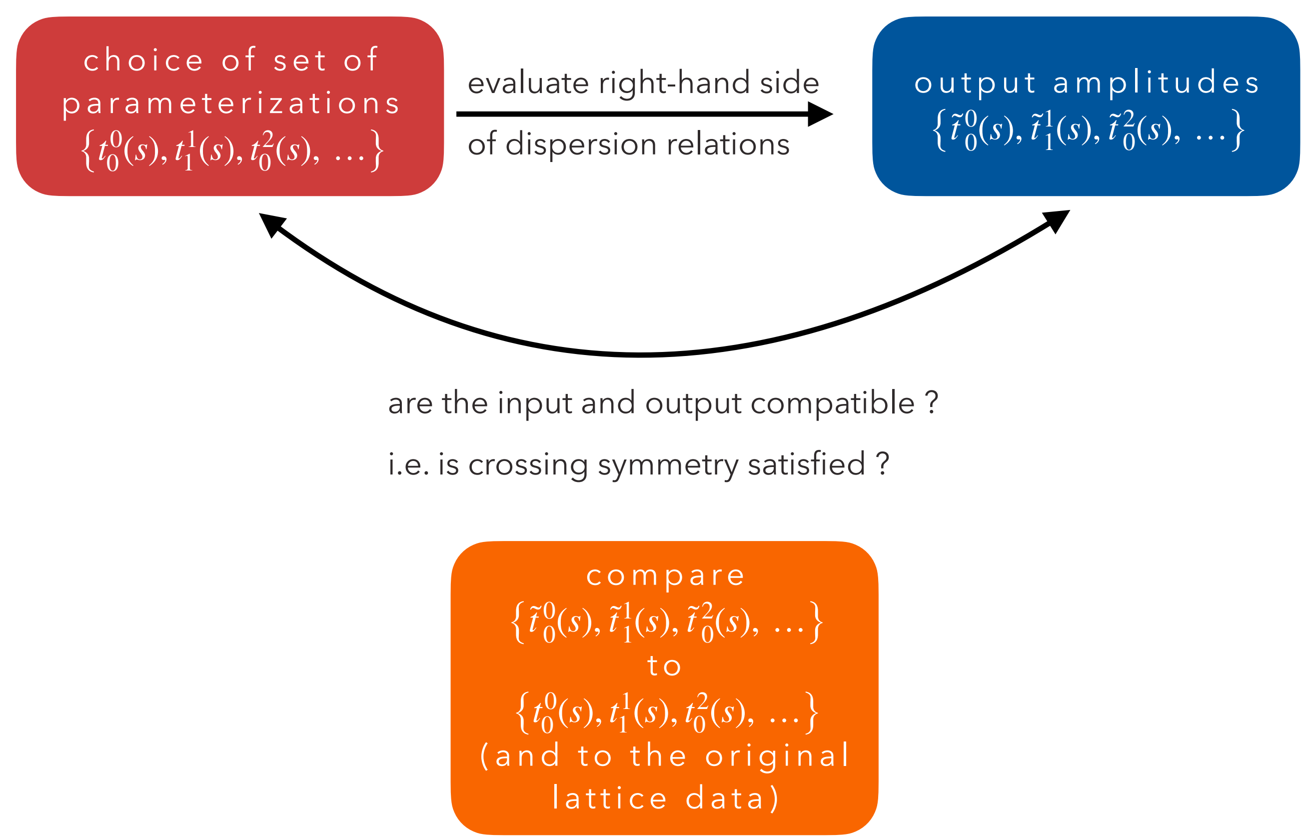
dispersive implementation of crossing

"twice subtracted" dispersion relations

$$\tilde{t}^{I}_{\ell}(s) = \tau^{I}_{\ell}(s) + \sum_{I',\ell'} \int_{s_{\mathrm{thr.}}}^{\infty} ds'\, K^{II'}_{\ell\ell'}(s',s)\, \mathrm{Im}\, t^{I'}_{\ell'}(s')$$

lattice QCD input: partial-wave amplitudes in $I = 0,1,2$

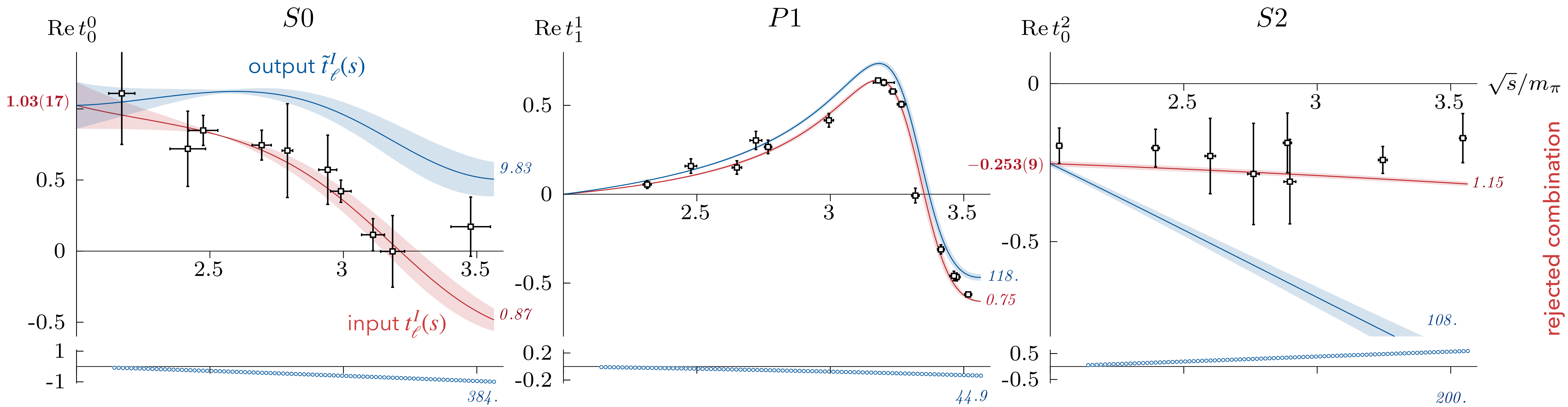
selects those combinations of parameterizations compatible with crossing



dispersion relation output at $m_\pi \sim 239\,\mathrm{MeV}$

$$\tilde{t}^I_\ell(s) = \tau^I_\ell(s) + \sum_{I',\ell'} \int_{s_{\mathrm{thr.}}}^{\infty} ds'\, K^{II'}_{\ell\ell'}(s',s)\, \mathrm{Im}\, t^{I'}_{\ell'}(s')$$

one particular choice:



visibly fails to satisfy the dispersion relations

this set of parameterizations **not compatible with crossing symmetry**

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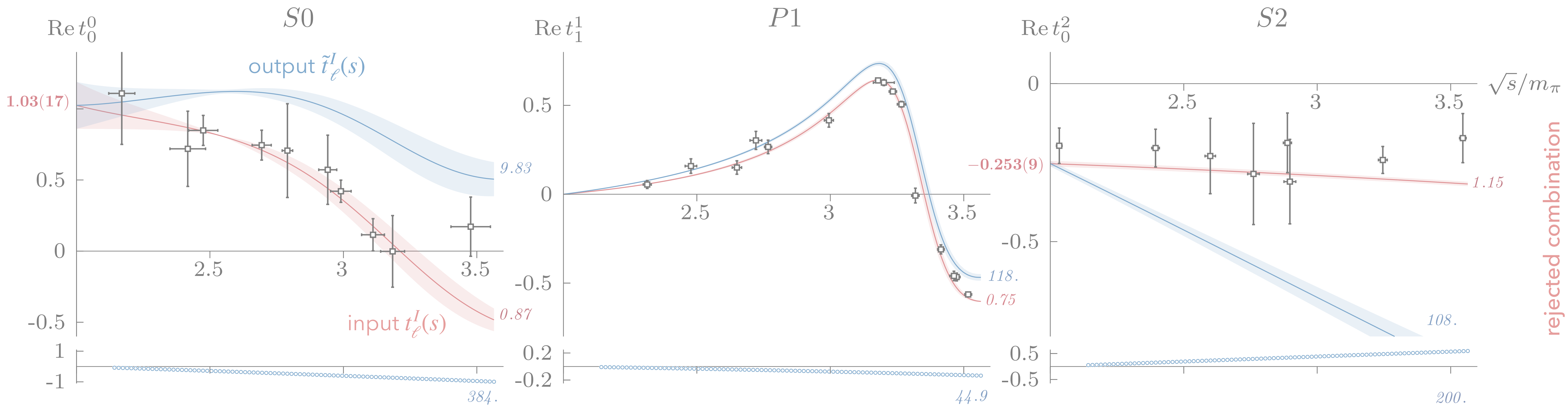
Arkaitz Rodas

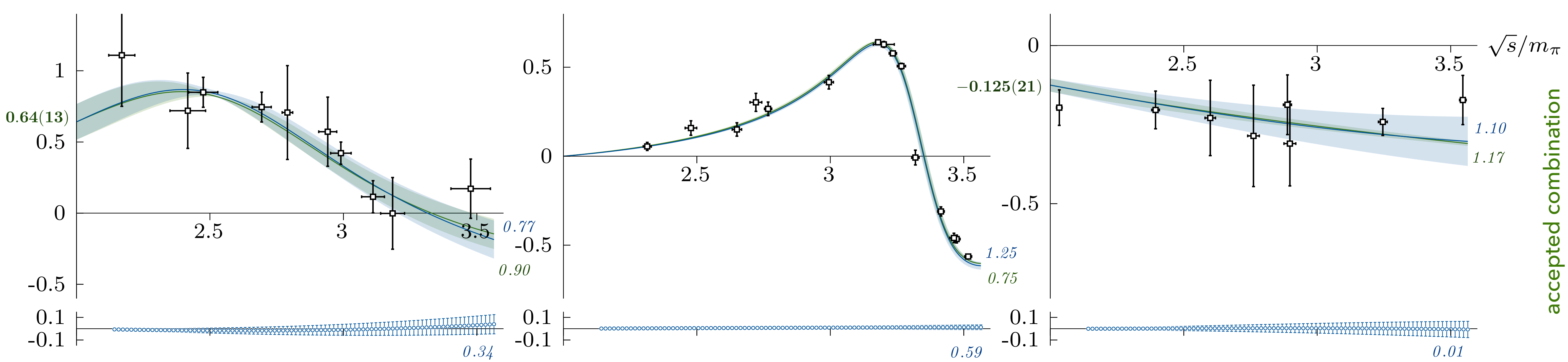
hadspec
'disperser-in-chief'

dispersion relation output at $m_\pi \sim 239\,\text{MeV}$

$$\tilde{t}^I_\ell(s) = \tau^I_\ell(s) + \sum_{I',\ell'} \int_{s_{\text{thr.}}}^{\infty} ds'\, K^{II'}_{\ell\ell'}(s',s)\, \text{Im}\, t^{I'}_{\ell'}(s')$$

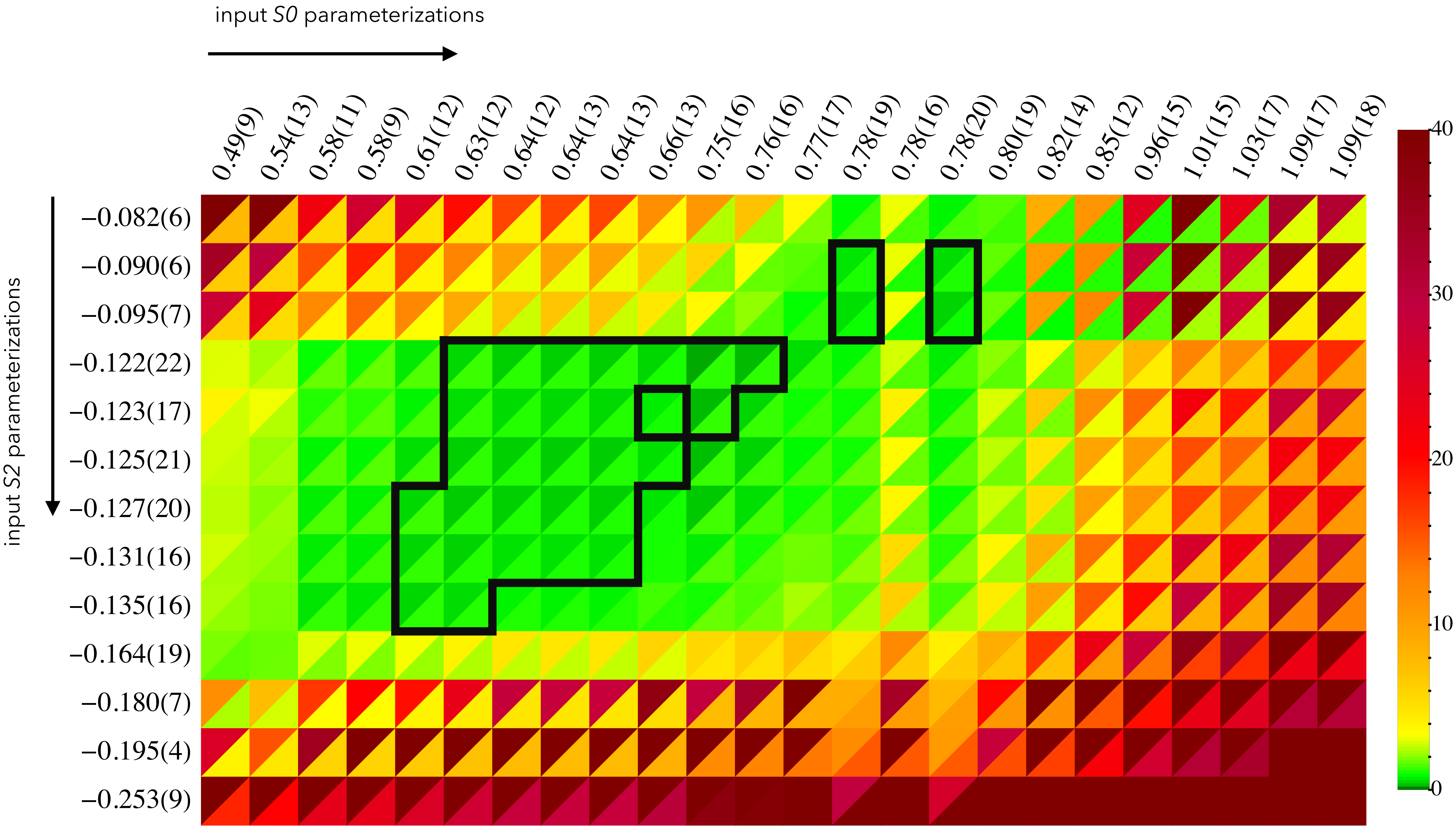


a different choice:

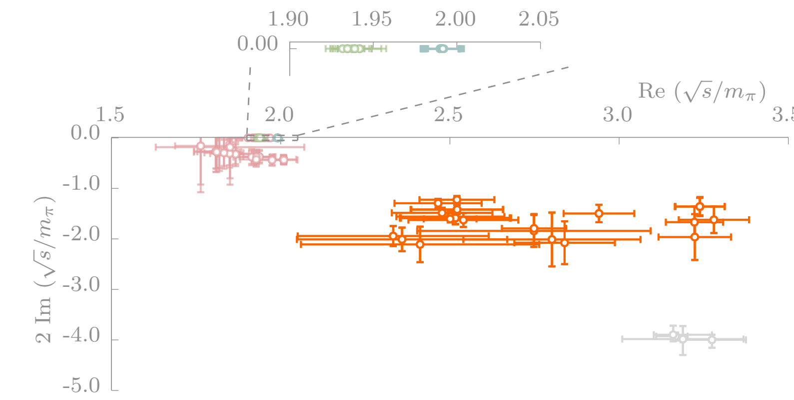


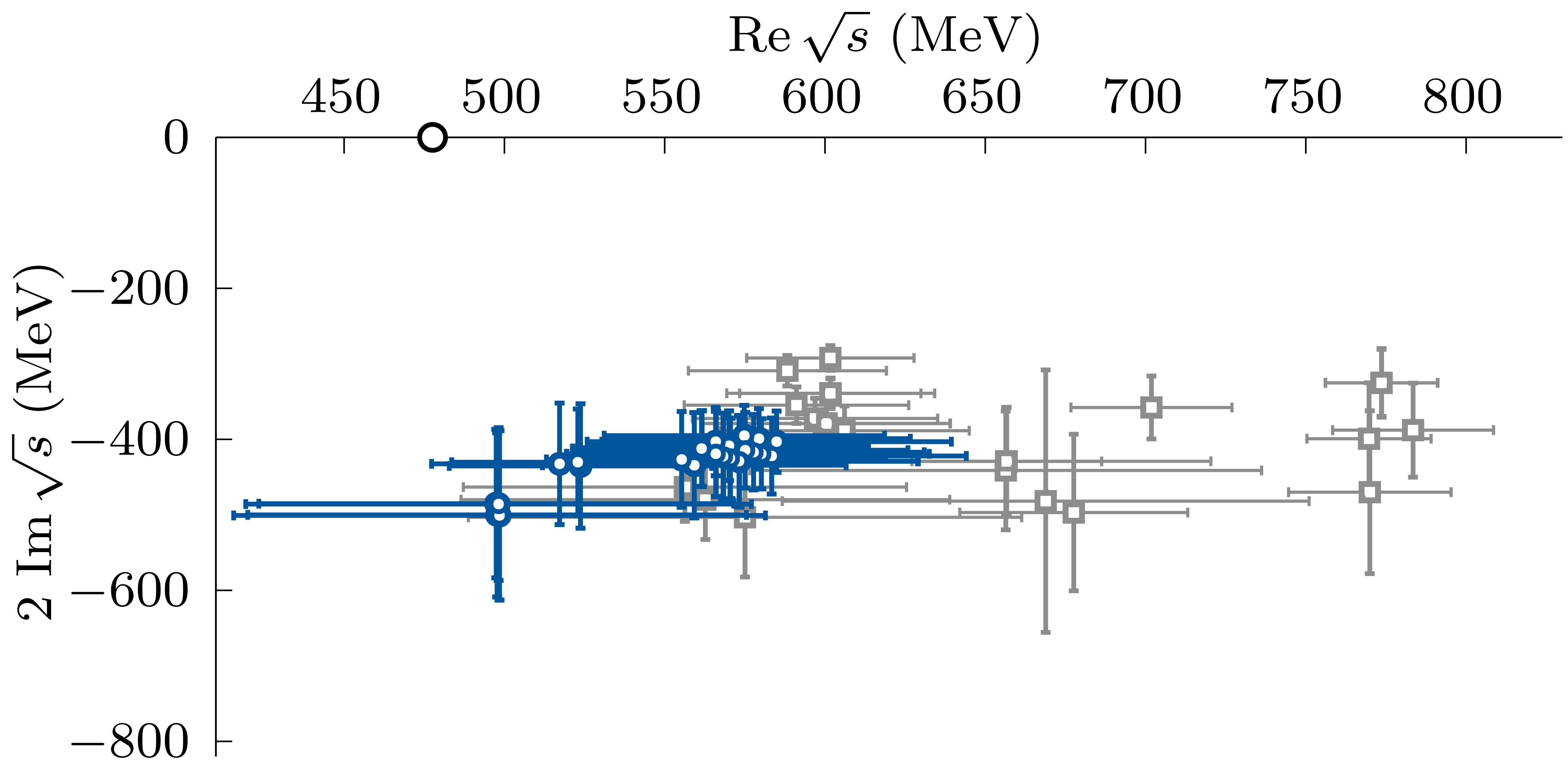
accepted/rejected combinations at $m_\pi \sim 239$ MeV

parameterizations labelled by their scattering length



σ resonance pole in dispersed amplitudes at $m_\pi \sim 239\,\mathrm{MeV}$





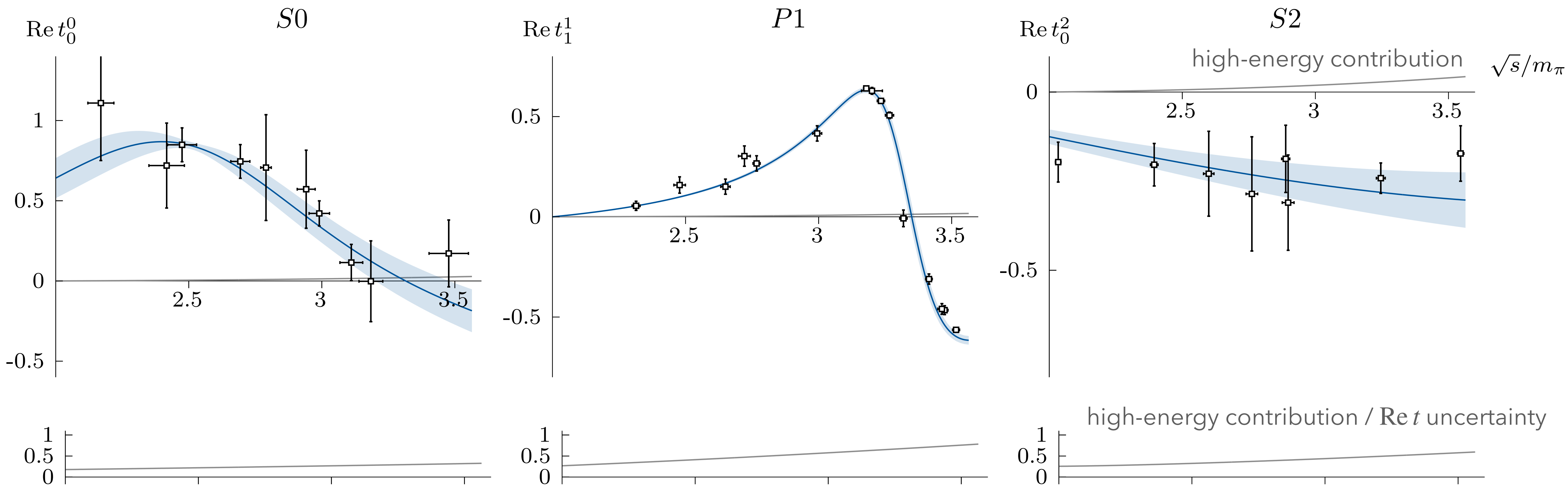
handling the high-energy part of the integral

$$\tilde{t}^I_\ell(s) = \tau^I_\ell(s) + \sum_{I',\ell'} \int_{s_{\text{thr.}}}^{\infty} ds'\, K^{II'}_{\ell\ell'}(s',s)\, \text{Im}\, t^{I'}_{\ell'}(s')$$

use a 'quark-mass scaled' Regge parameterization …

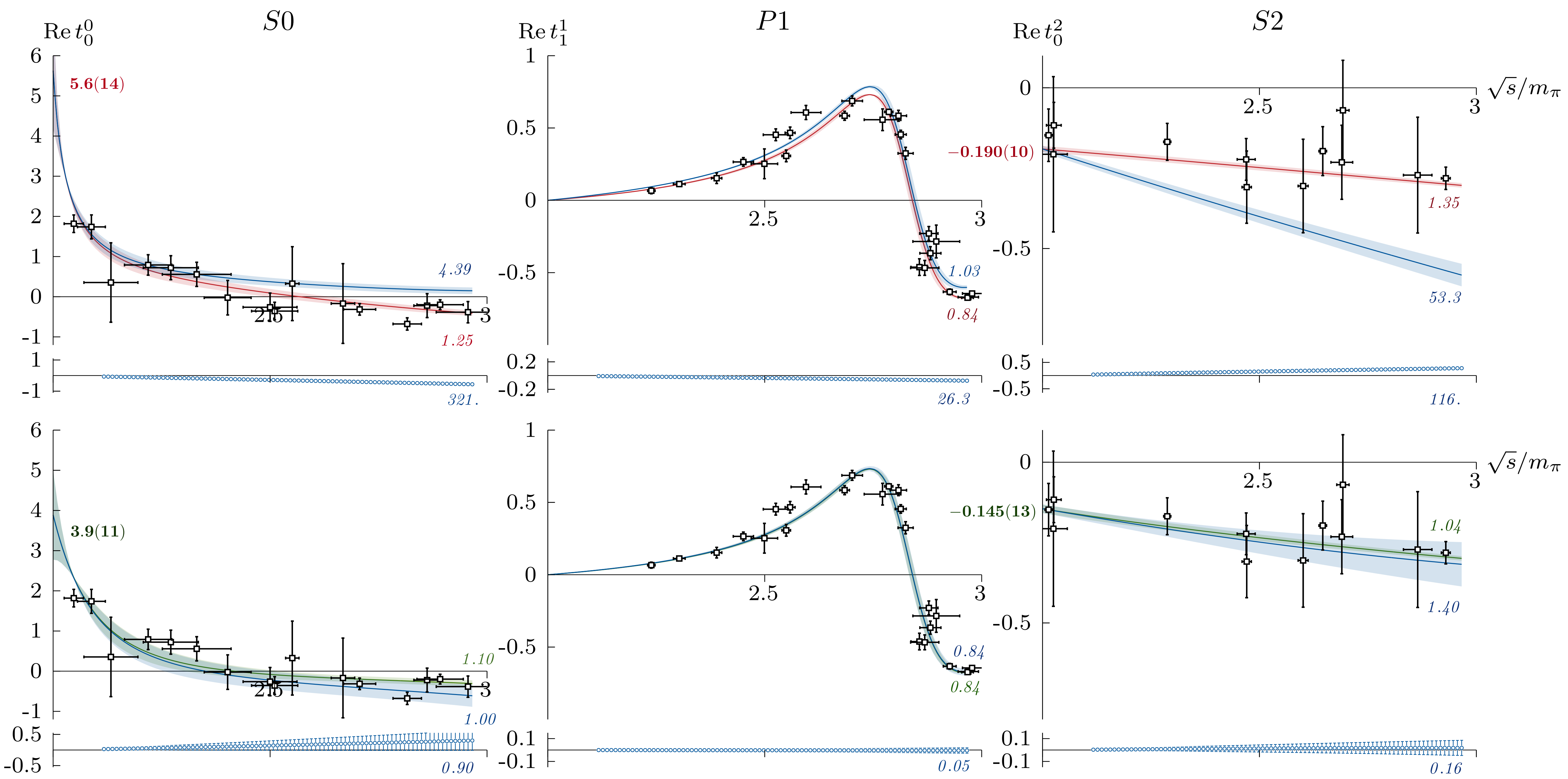
… but observe that results are largely insensitive to this:

$$\text{Re}\, \tilde{t}^I_\ell(s)\,\big|_{\text{Regge}} = \sum_{I',\ell'} \int_{s_{\text{high}}}^{\infty} ds'\, \text{Re}\, K^{II'}_{\ell\ell'}(s',s)\, \text{Im}\, t^{I'}_{\ell'}(s')\Big|_{\text{Regge}}$$

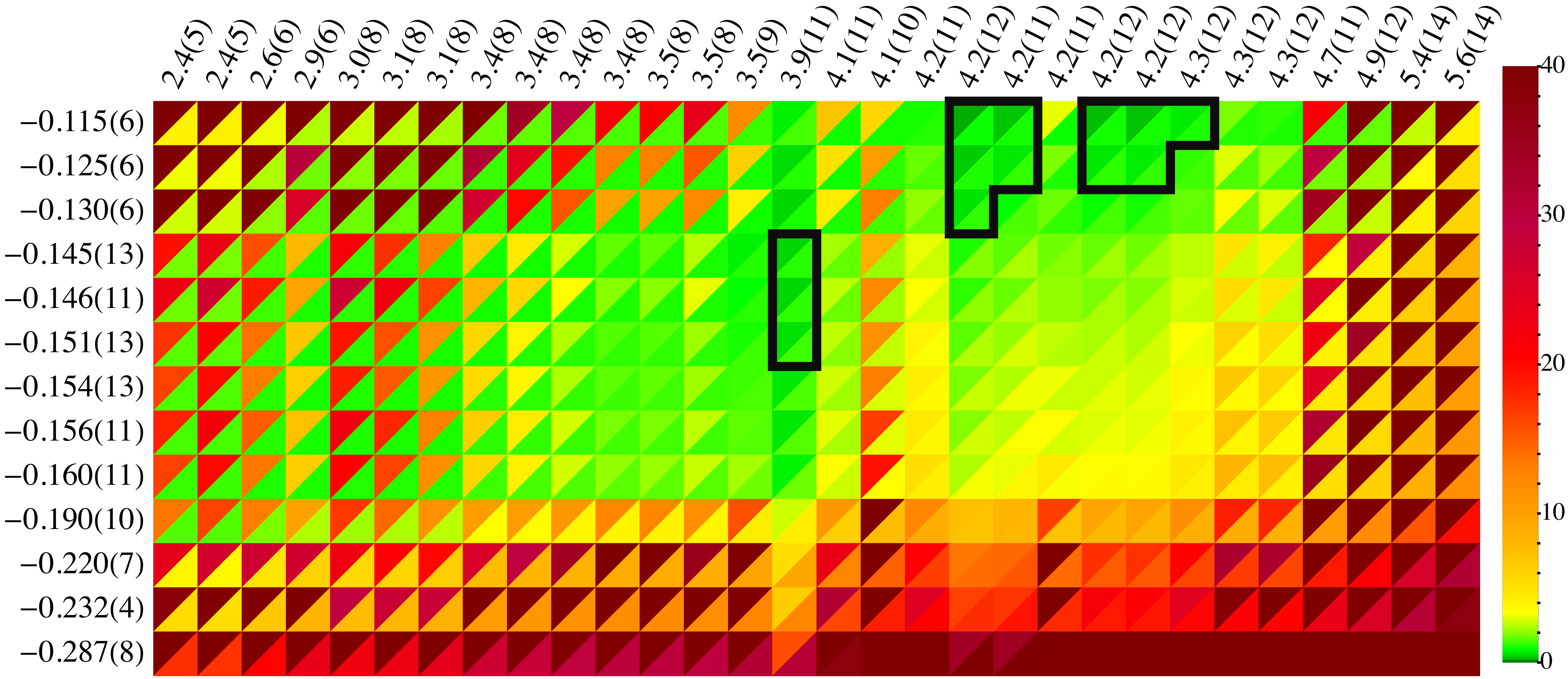


impact of high-energy part smaller than the statistical uncertainty what about another pion mass …

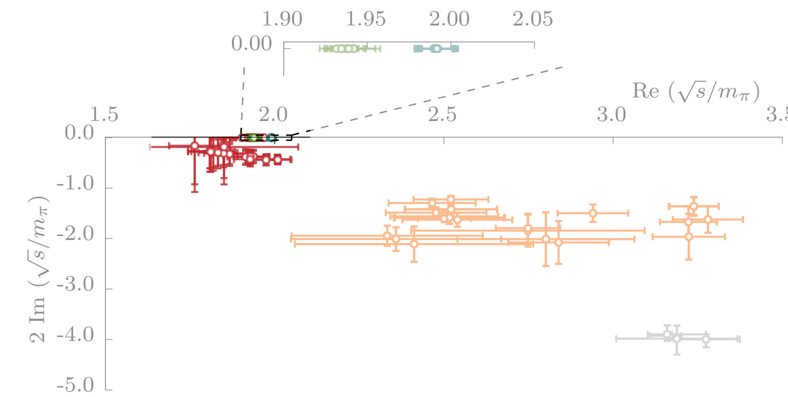
dispersion relation output at $m_\pi \sim 283\,\mathrm{MeV}$

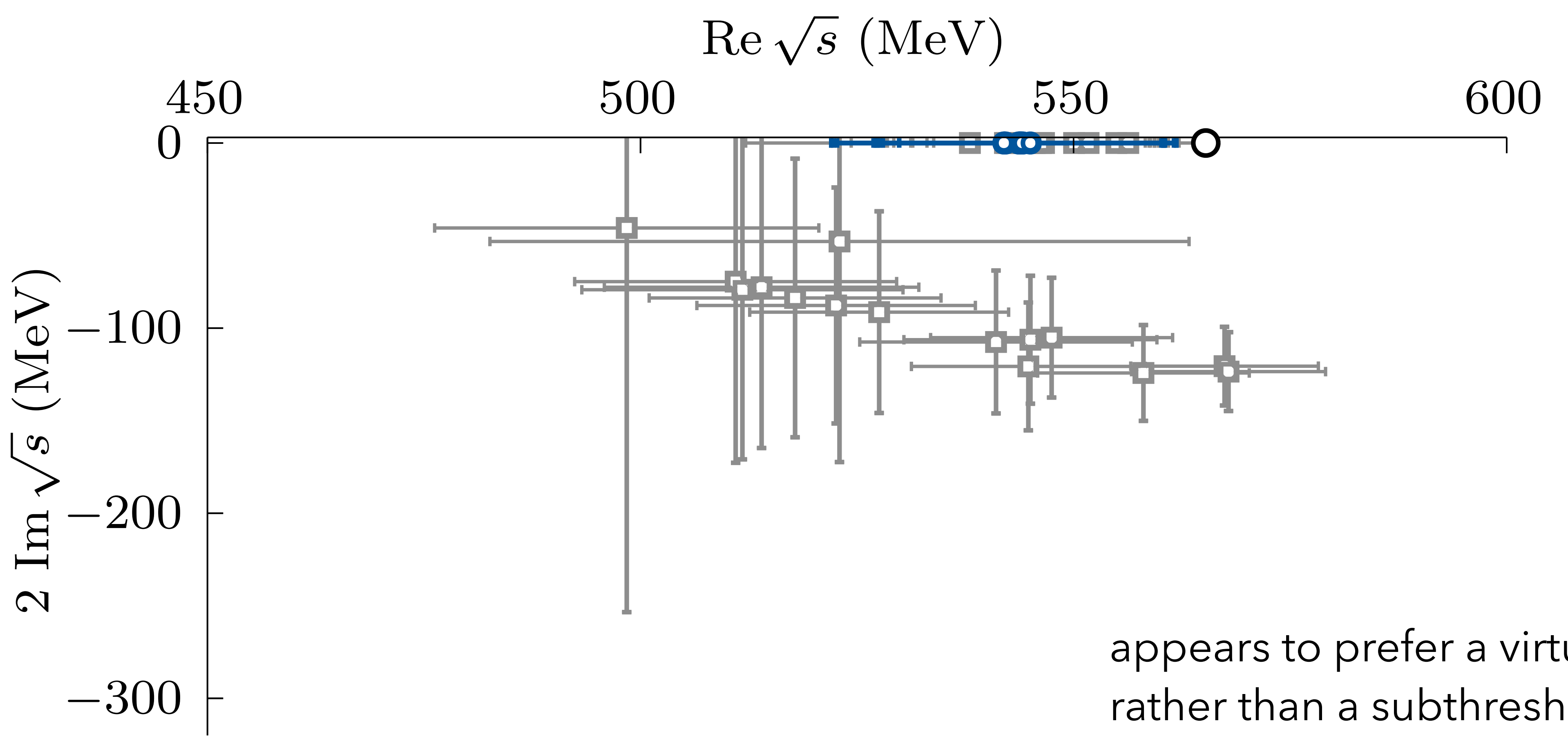


accepted/rejected combinations at $m_\pi \sim 283\,\mathrm{MeV}$



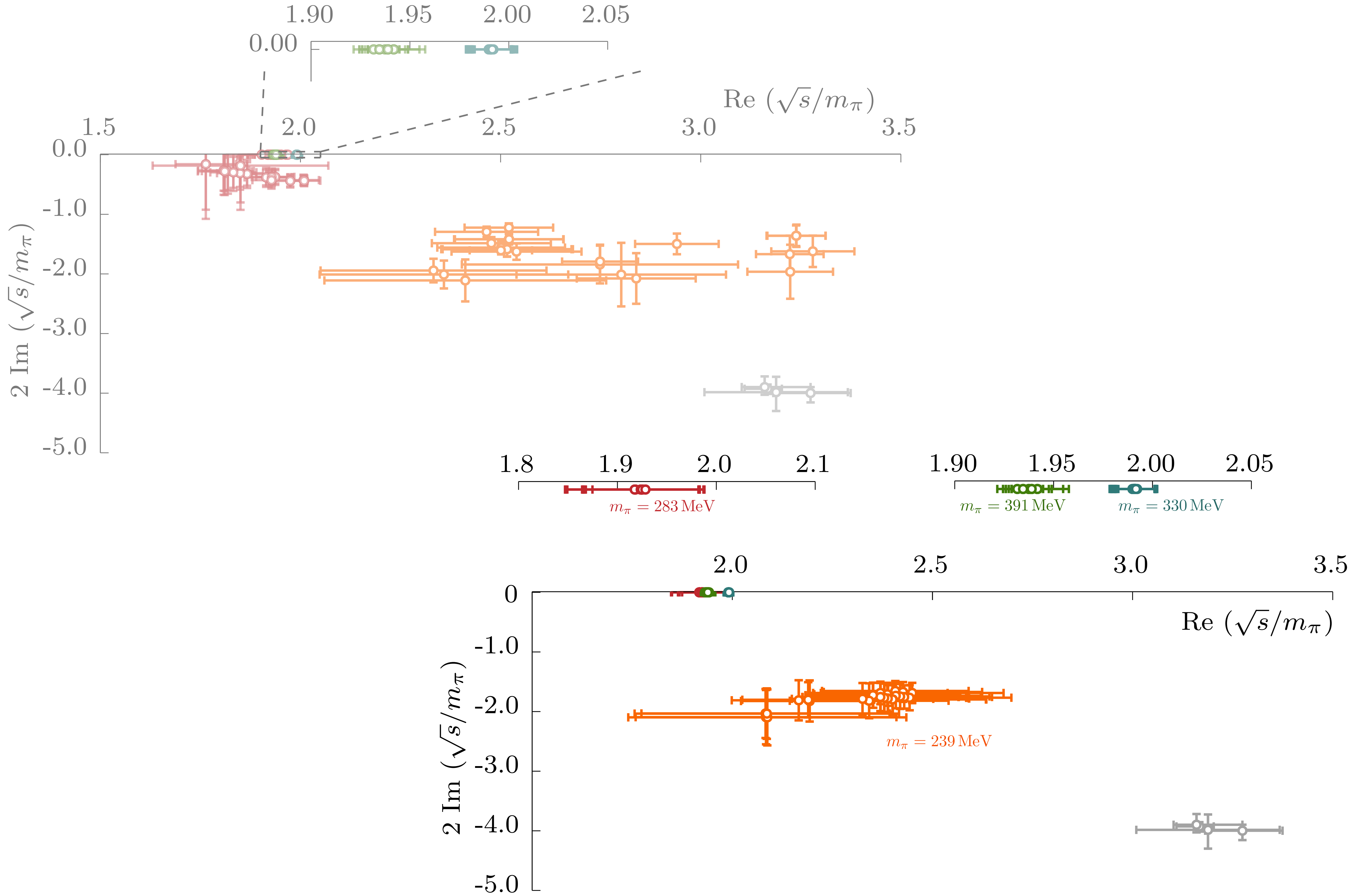
σ pole in dispersed amplitudes at $m_\pi \sim 283$ MeV



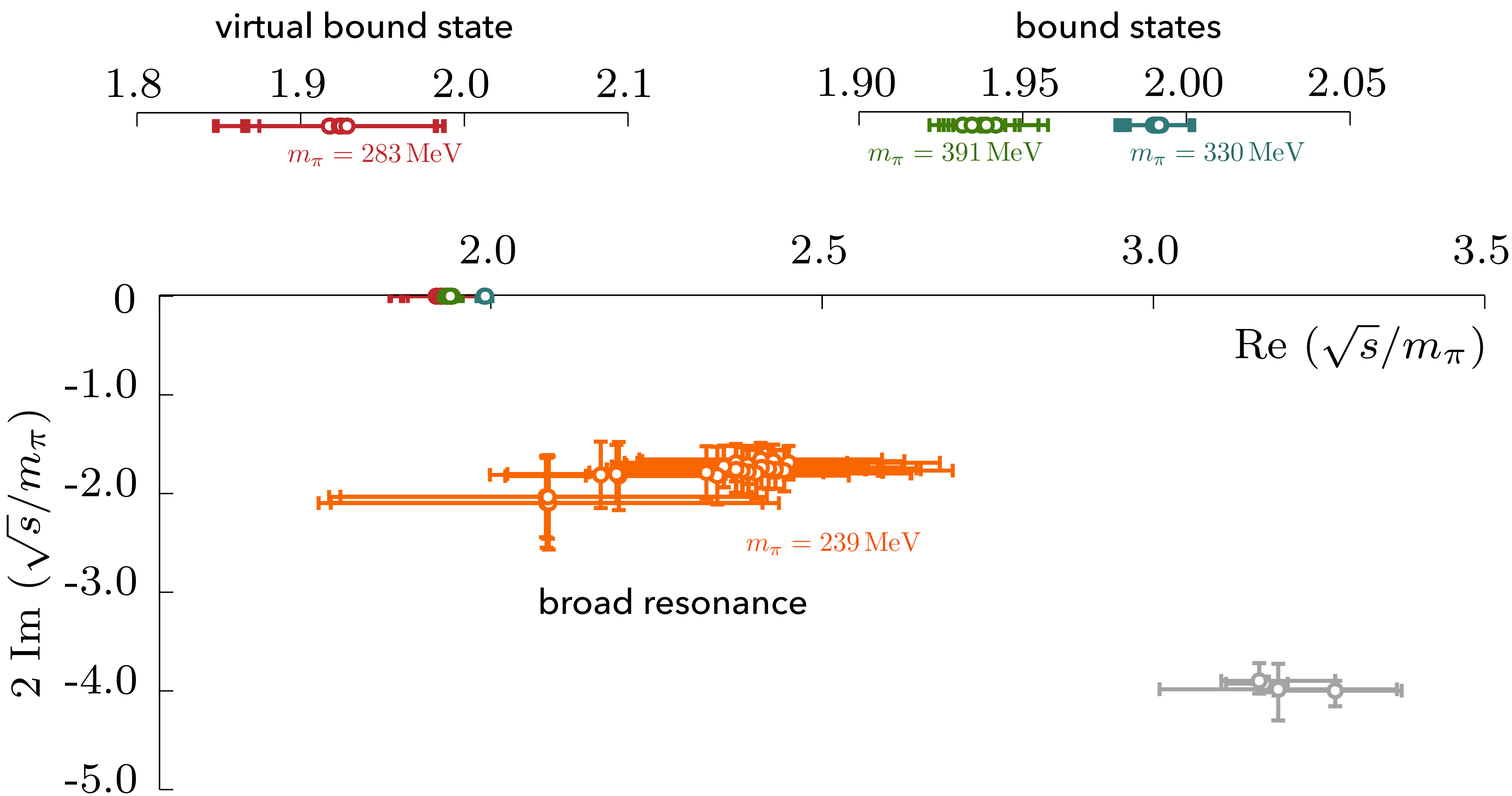


appears to prefer a virtual bound state
rather than a subthreshold resonance

impact of imposing crossing symmetry on σ pole determination



quark mass evolution of σ pole



bound state must hit threshold between $m_\pi \sim 330\,\mathrm{MeV}$ and $m_\pi \sim 283\,\mathrm{MeV}$, moving onto second sheet as a virtual bound state

arrival of another second sheet pole, and the move off into the complex plane below $m_\pi \sim 283\,\mathrm{MeV}$ but above $m_\pi \sim 239\,\mathrm{MeV}$

narrow resonances – insensitive to parameterization details

broad, low-energy resonances – *potentially* sensitive to left-hand cut details

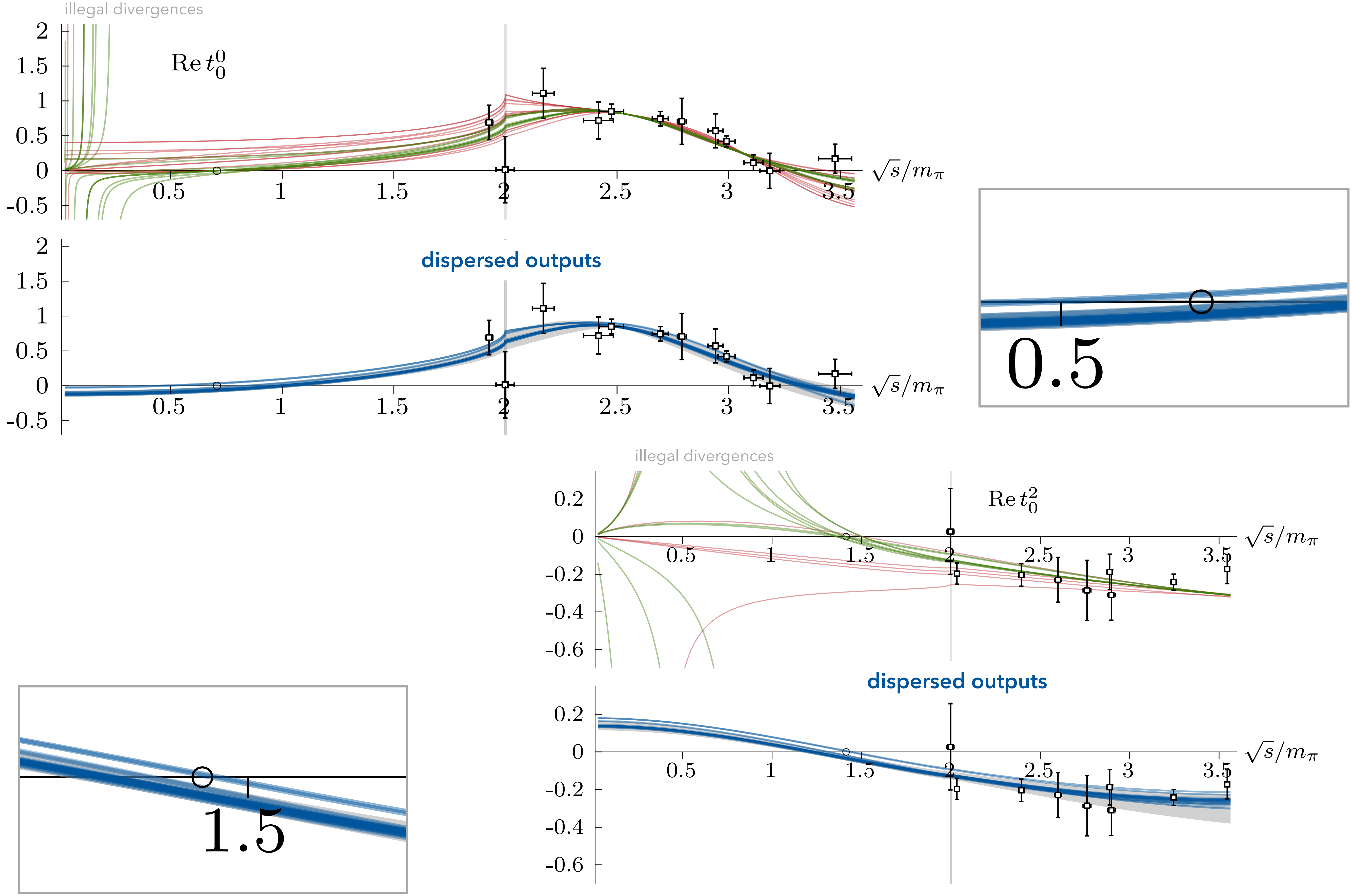
proves to be the case for the σ in elastic $\pi\pi$

fortunate in the case of $\pi\pi$ that **crossing symmetry** can be used practically via **dispersion relations** applied to lattice-origin amplitudes at relatively low pion masses for the first time

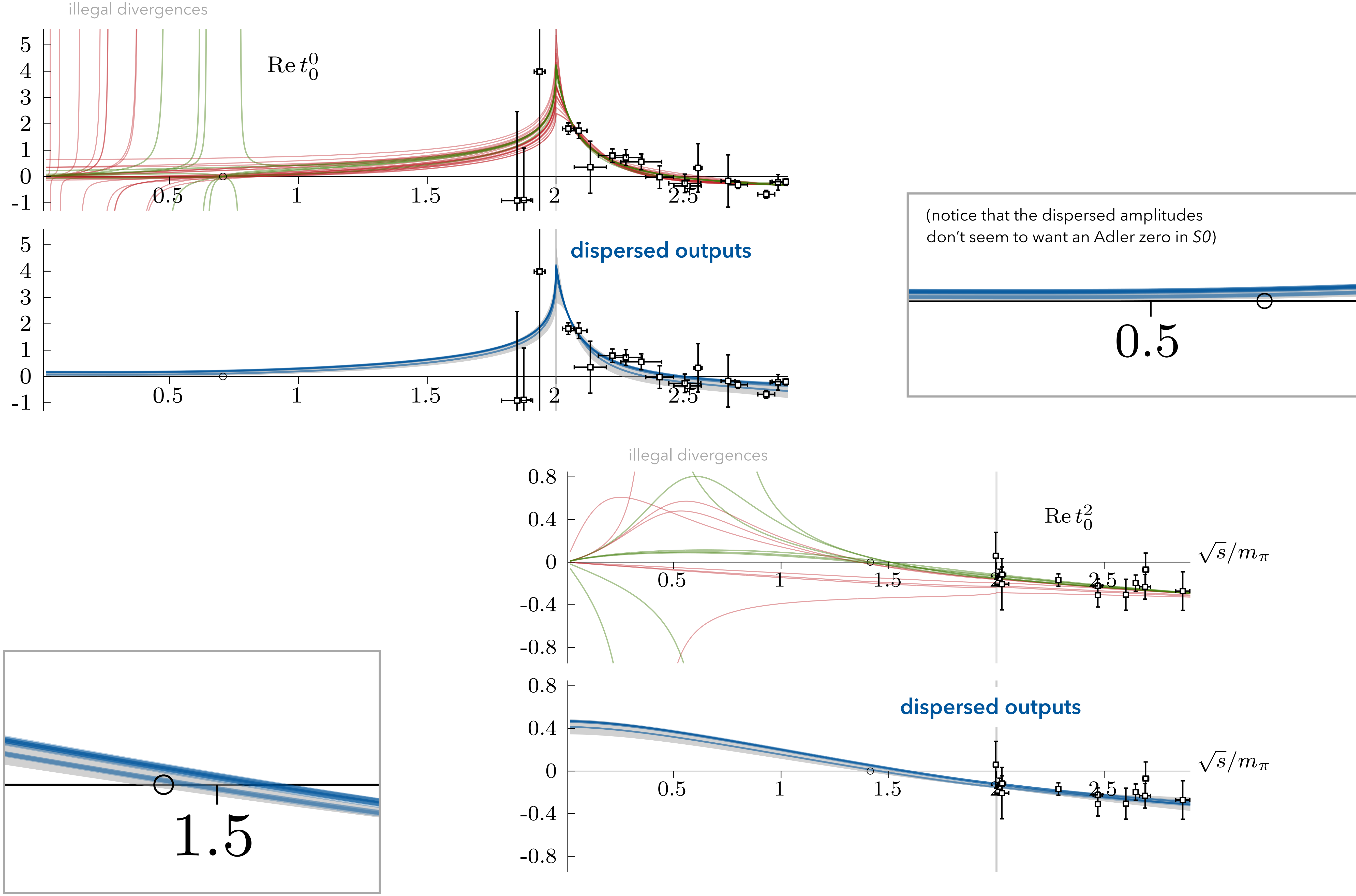
check that different partial-waves are **together** compatible with crossing

not enough time to show it here, but role of (Adler) subthreshold zeroes explored in relatively model-independent manner in this study

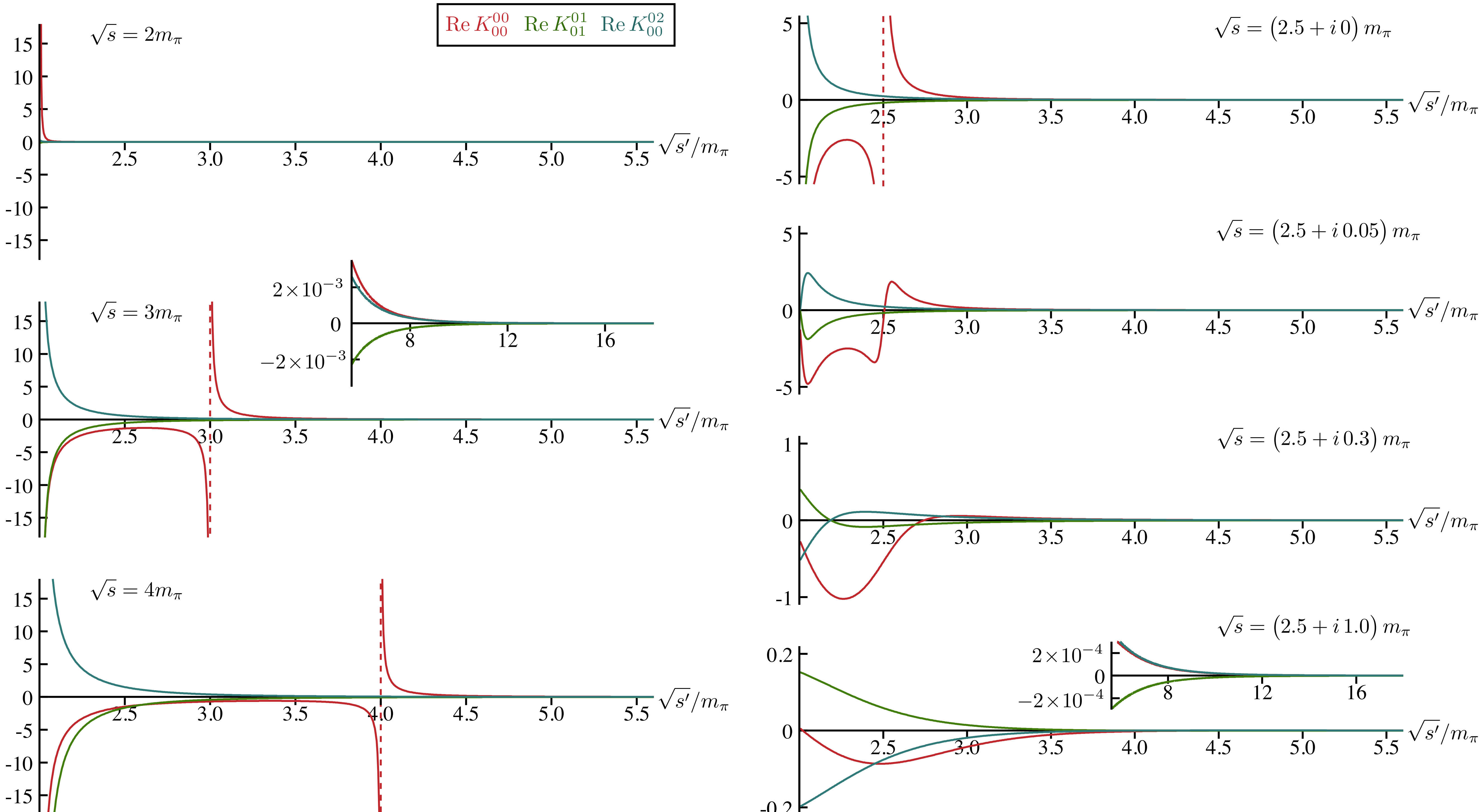
dispersed amplitudes are constrained subthreshold at $m_\pi \sim 239\,\mathrm{MeV}$



dispersed amplitudes are constrained subthreshold at $m_\pi \sim 283$ MeV



the dispersive kernels $K^{II'}_{\ell\ell'}(s,s')$ – twice subtracted



notice the sensitivity to low energy scattering in other partial-waves at the σ pole location

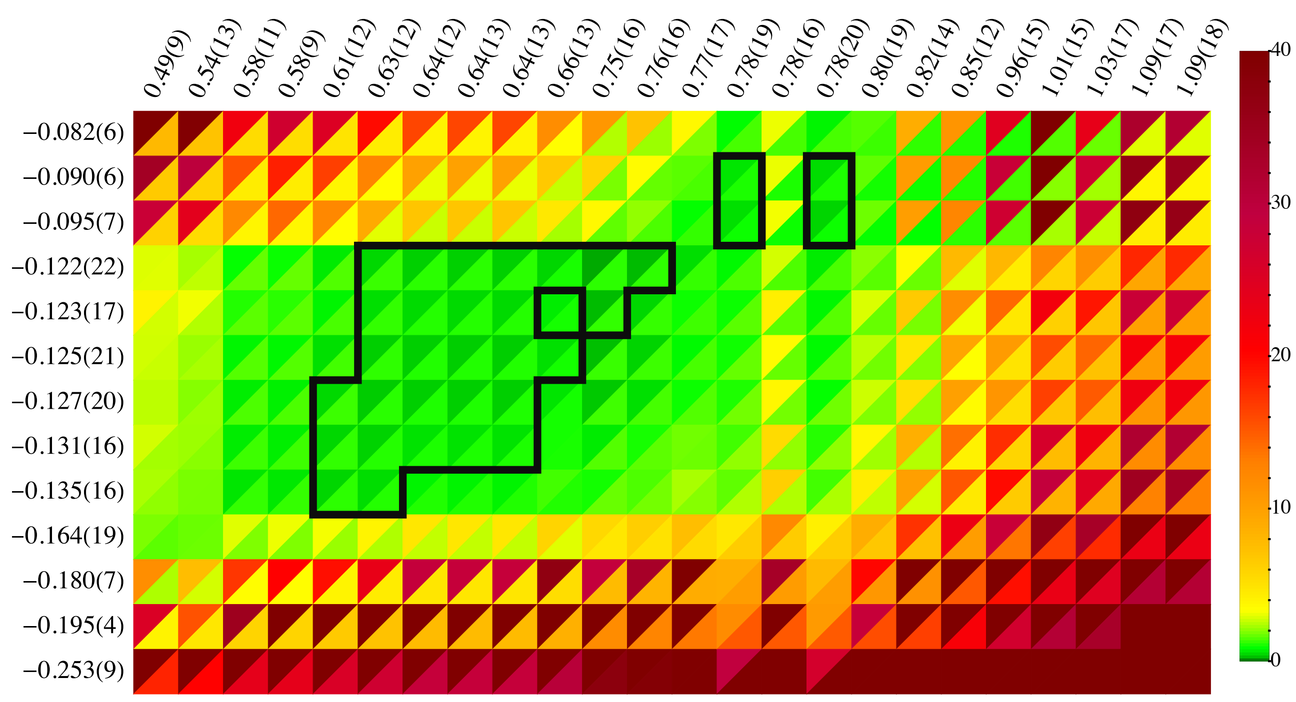
numerical comparisons

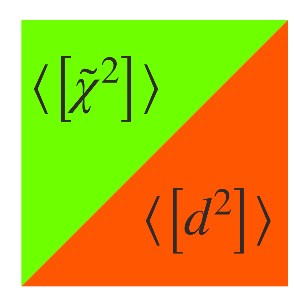
compare to the 'lattice data' $\left[\tilde{\chi}^2\right]^I_\ell \equiv \sum_{i,j=1}^{N_{\text{lat}}} \left(\frac{\mathfrak{f}_i - \tilde{f}^I_\ell(s_i)}{\Delta_i}\right) \text{corr}(\mathfrak{f}_i, \mathfrak{f}_j)^{-1} \left(\frac{\mathfrak{f}_j - \tilde{f}^I_\ell(s_j)}{\Delta_j}\right)$

compare to the input amplitude $\left[d^2\right]^I_\ell \equiv \sum_{i=1}^{N_{\text{smpl}}} \left(\frac{\tilde{f}^I_\ell(s_i) - f^I_\ell(s_i)}{\Delta\left[\tilde{f}^I_\ell(s_i) - f^I_\ell(s_i)\right]}\right)^2$

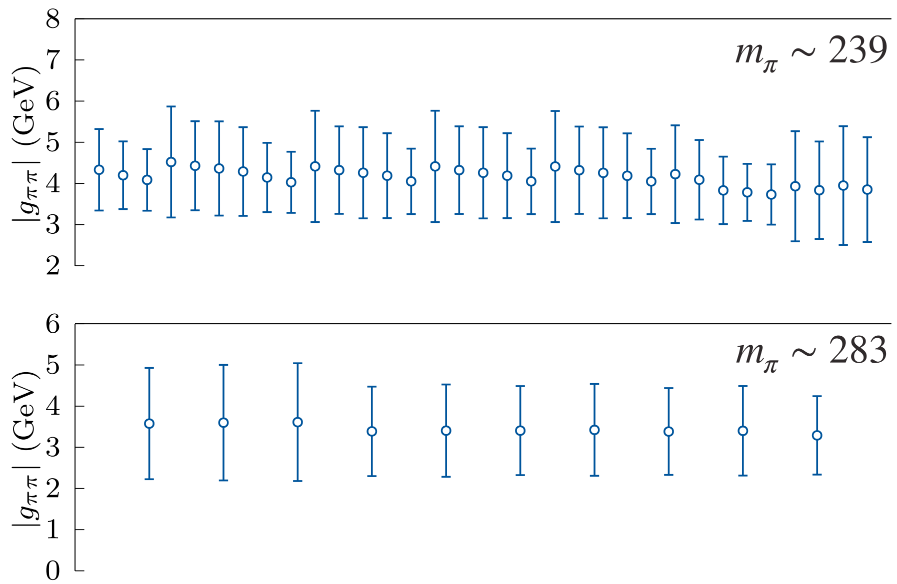
sampling amplitudes at a large but finite number of points

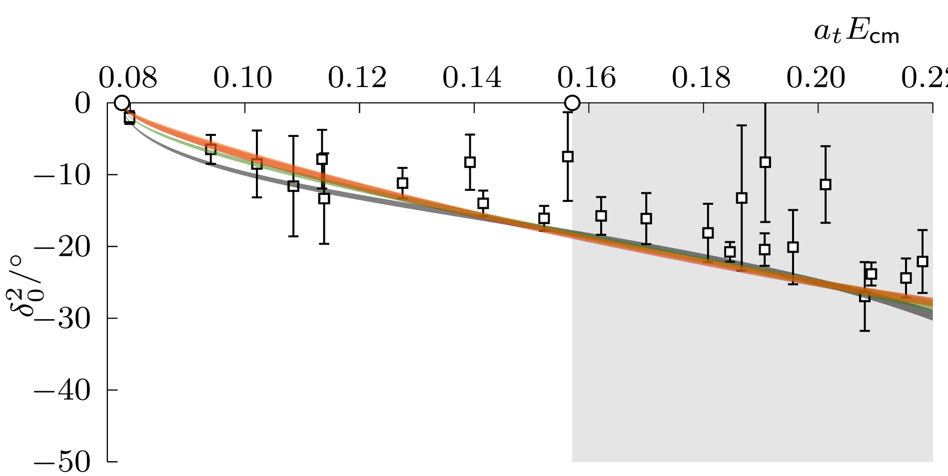
apply some cuts to value of each
(arbitrary, subjective, but we explored the sensitivity to the cut value)

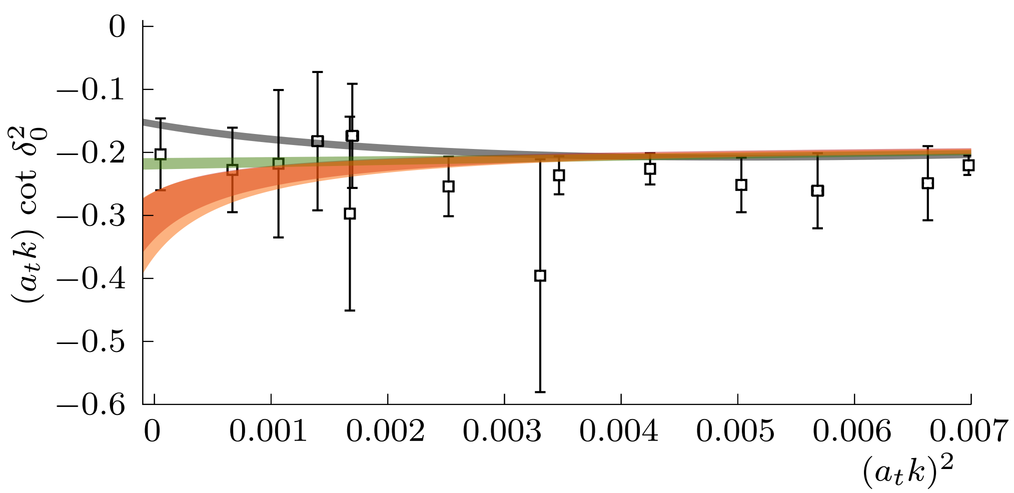


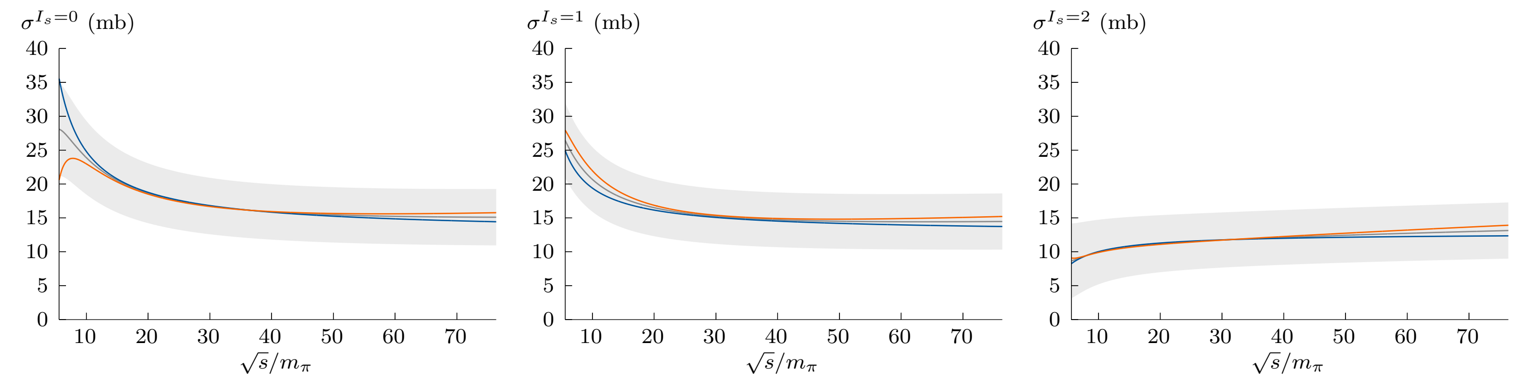


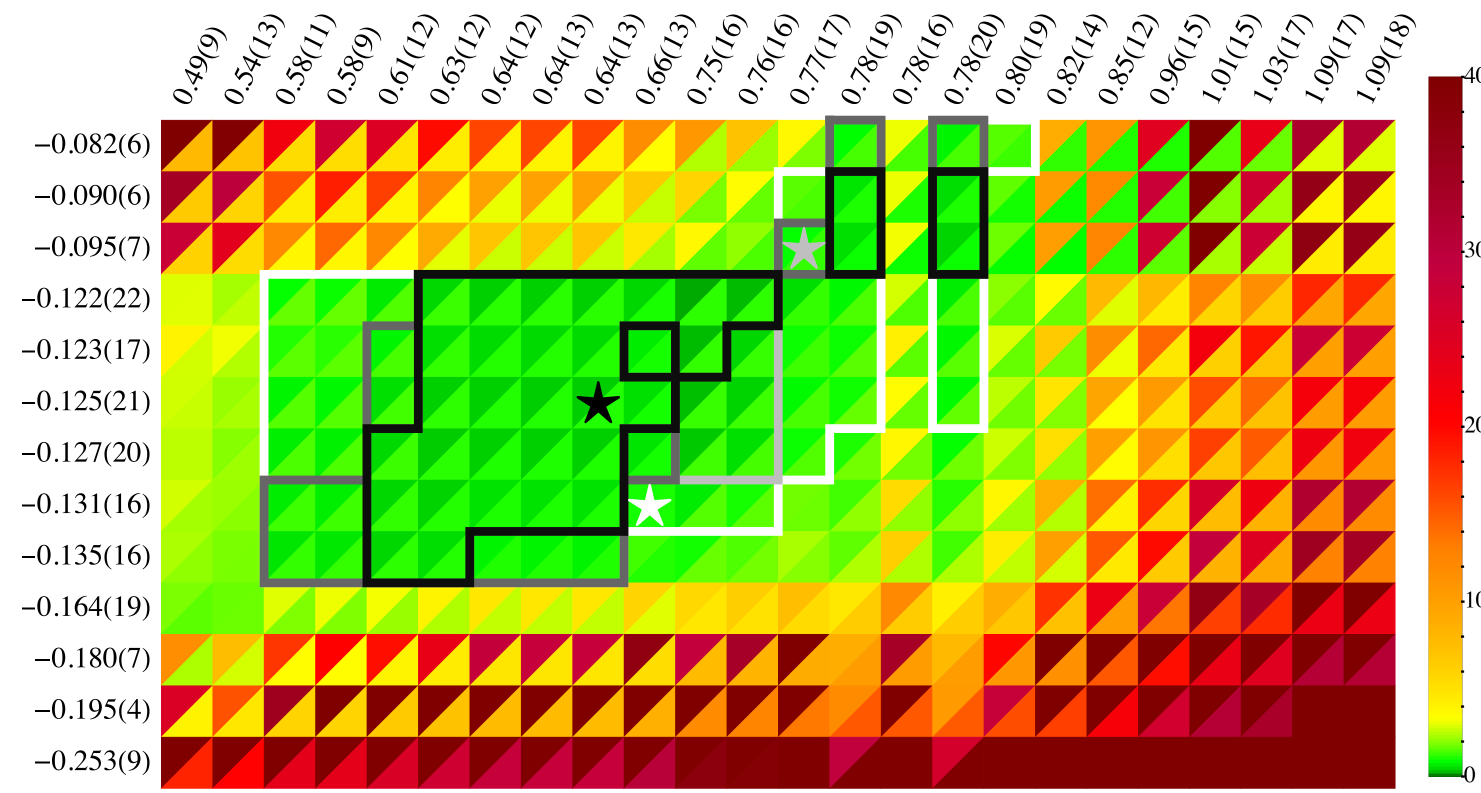
the dusty storage room of unlabelled plots

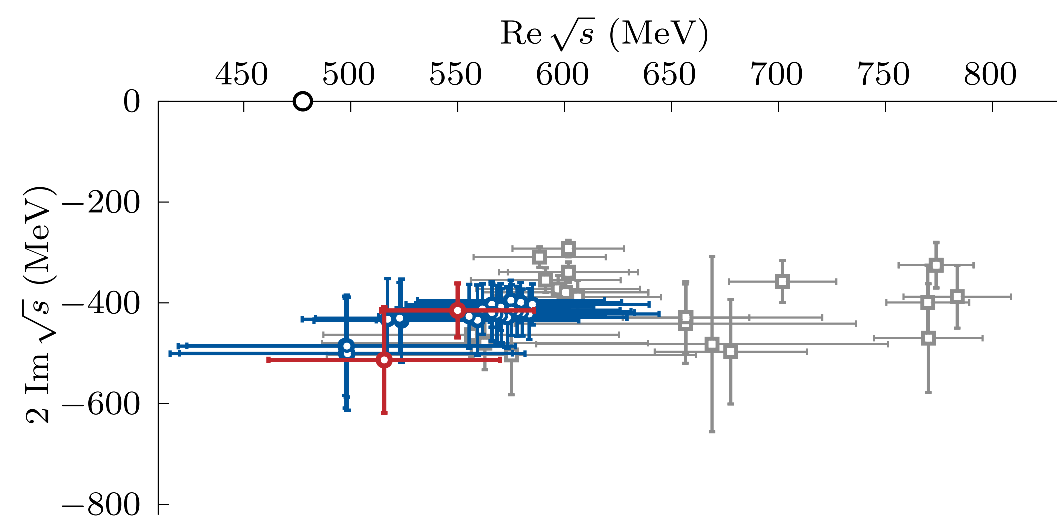




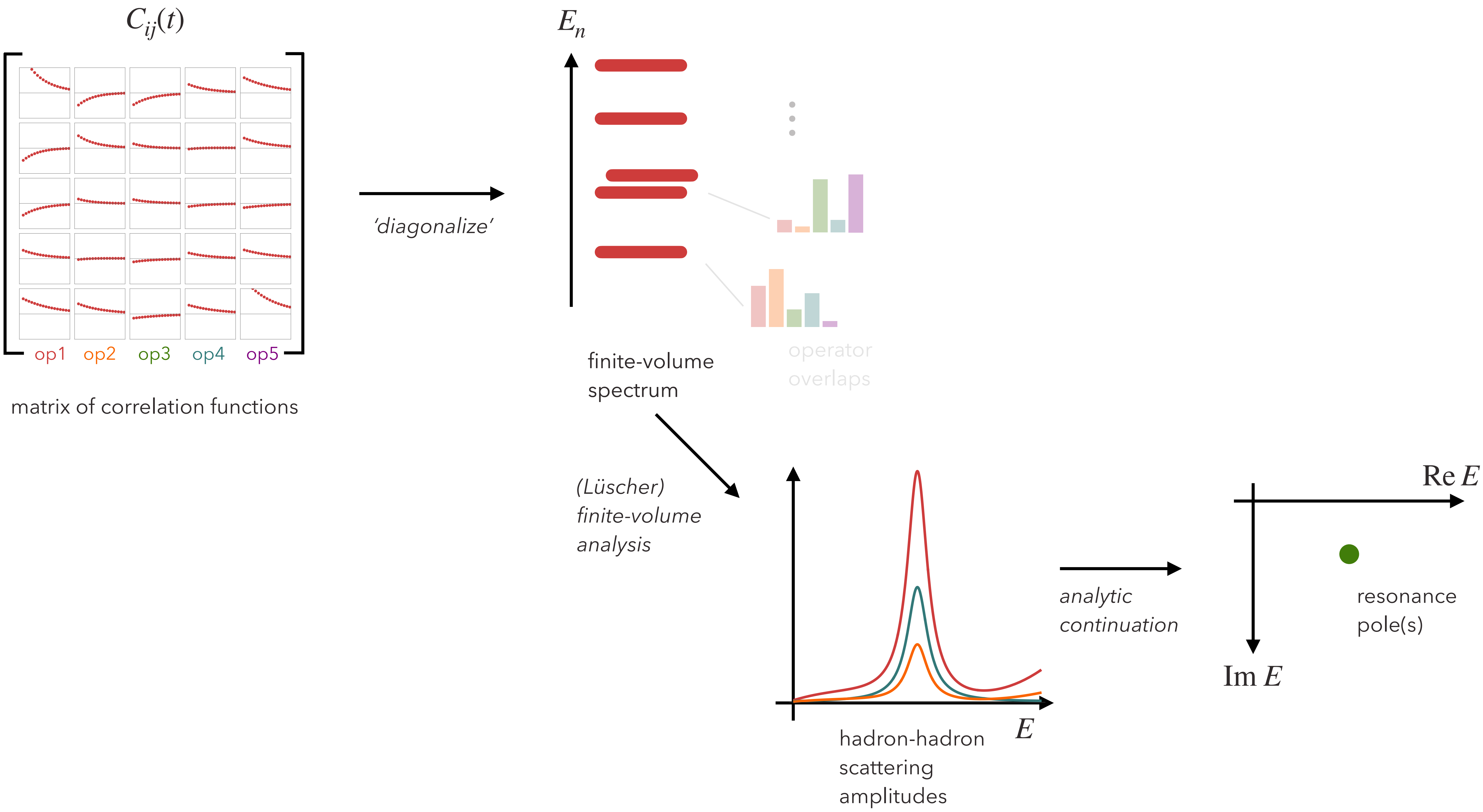








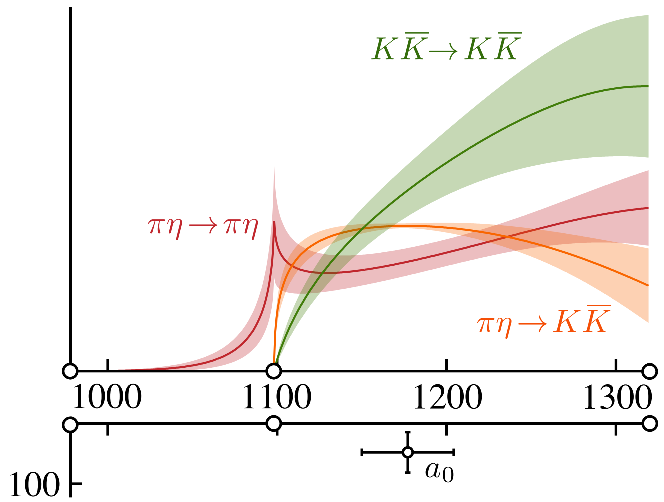
lattice QCD approach to scattering & resonances – the cartoon



lattice QCD approach to scattering & resonances

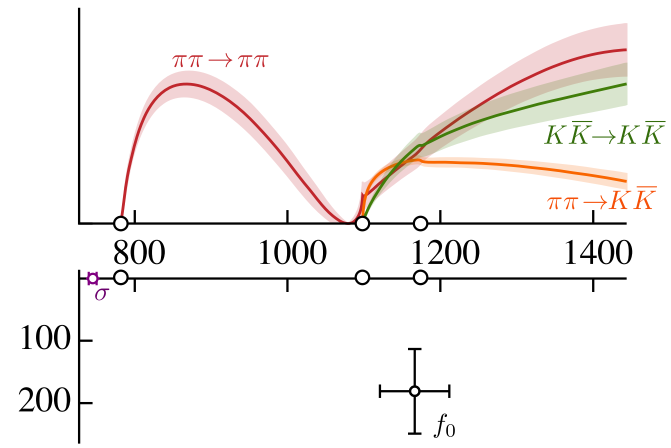
quite a few cases explored by *hadspec* (at unphysical pion masses), just a sample …

$J^P = 0^+ \; I^G = 1^- \; (\eta\pi, K\bar{K})$

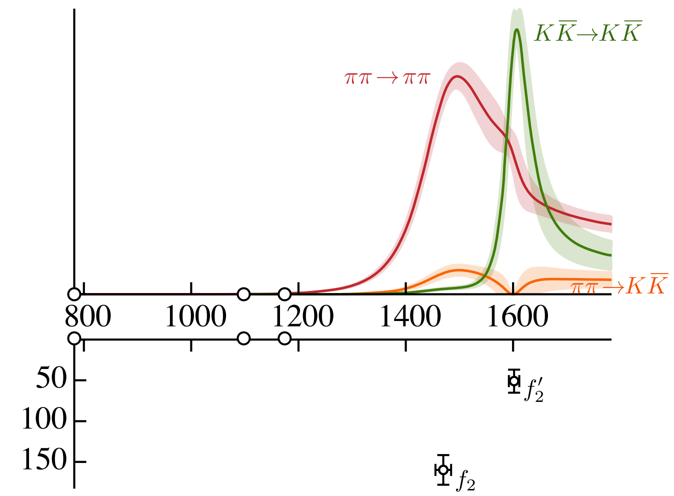


PRD 93 094506 (2016)

$J^P = 0^+ \; I^G = 0^+ \; (\pi\pi, K\bar{K}, \eta\eta)$

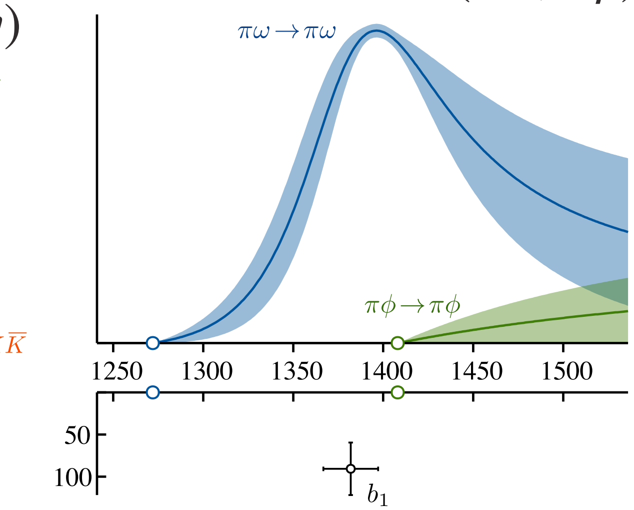


$J^P = 2^+ \; I^G = 0^+ \; (\pi\pi, K\bar{K}, \eta\eta)$



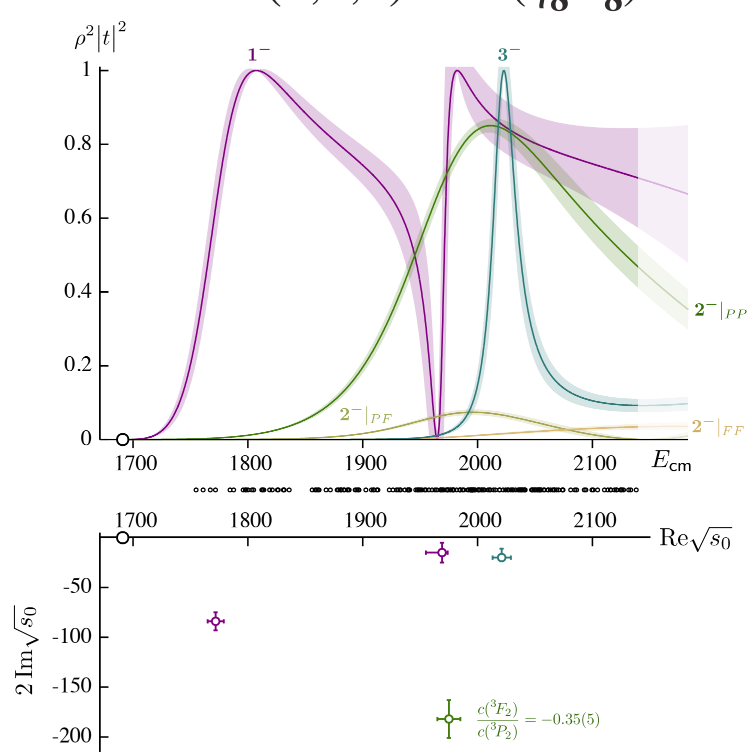
PRD 97 054513 (2018)

$J^P = 1^+ \; I^G = 1^- \; (\pi\omega, \pi\phi)$



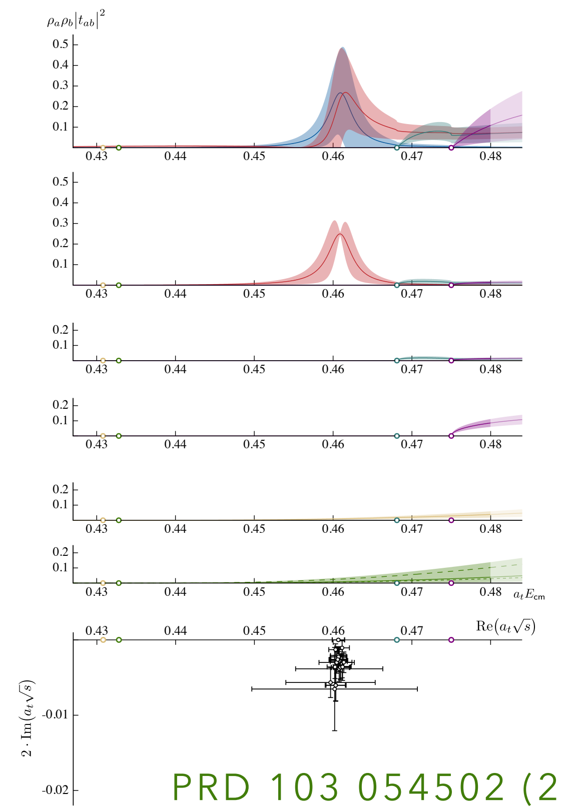
PRD 100 054506 (2019)

$J^{PC} = (1{,}2{,}3)^{--} \; \mathbf{1} \; (\eta_8\omega_8)$



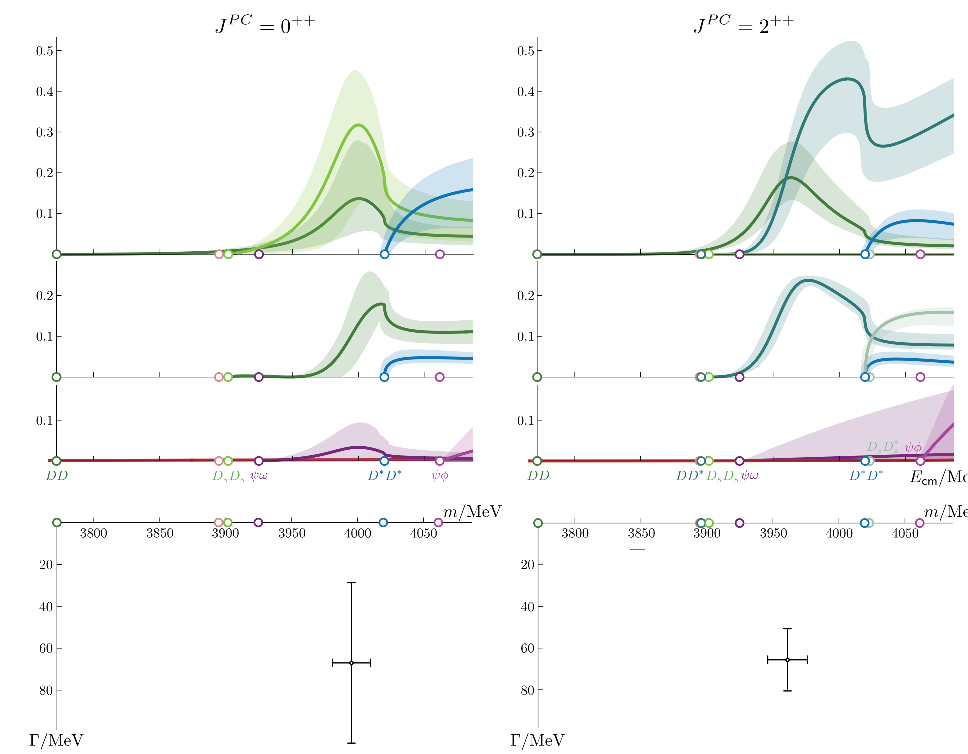
PRD 103 074502 (2021)

$J^{PC} = 1^{-+} \; \mathbf{8} \; (\eta_8\eta_1, \eta_8\omega_8, \eta_8 h_{18}, \eta_8 f_{18}, \ldots)$



PRD 103 054502 (2021)

$J^{PC} = (0{,}2)^{++} \; c\bar{c} \; (D\bar{D}, D\bar{D}^*, D_s\bar{D}_s, \ldots)$



PRL 132 241901 (2024)

σ pole in unitarized chiral perturbation theory

J.R. Pelaez ...

PRD81 054035 (2010)

resonance becomes a **virtual bound state** near m_π~350 MeV ...

... then a **bound state** near m_π~420 MeV

"the exact m_π value when this happens is not very reliable"