Lattice Gauge Theory

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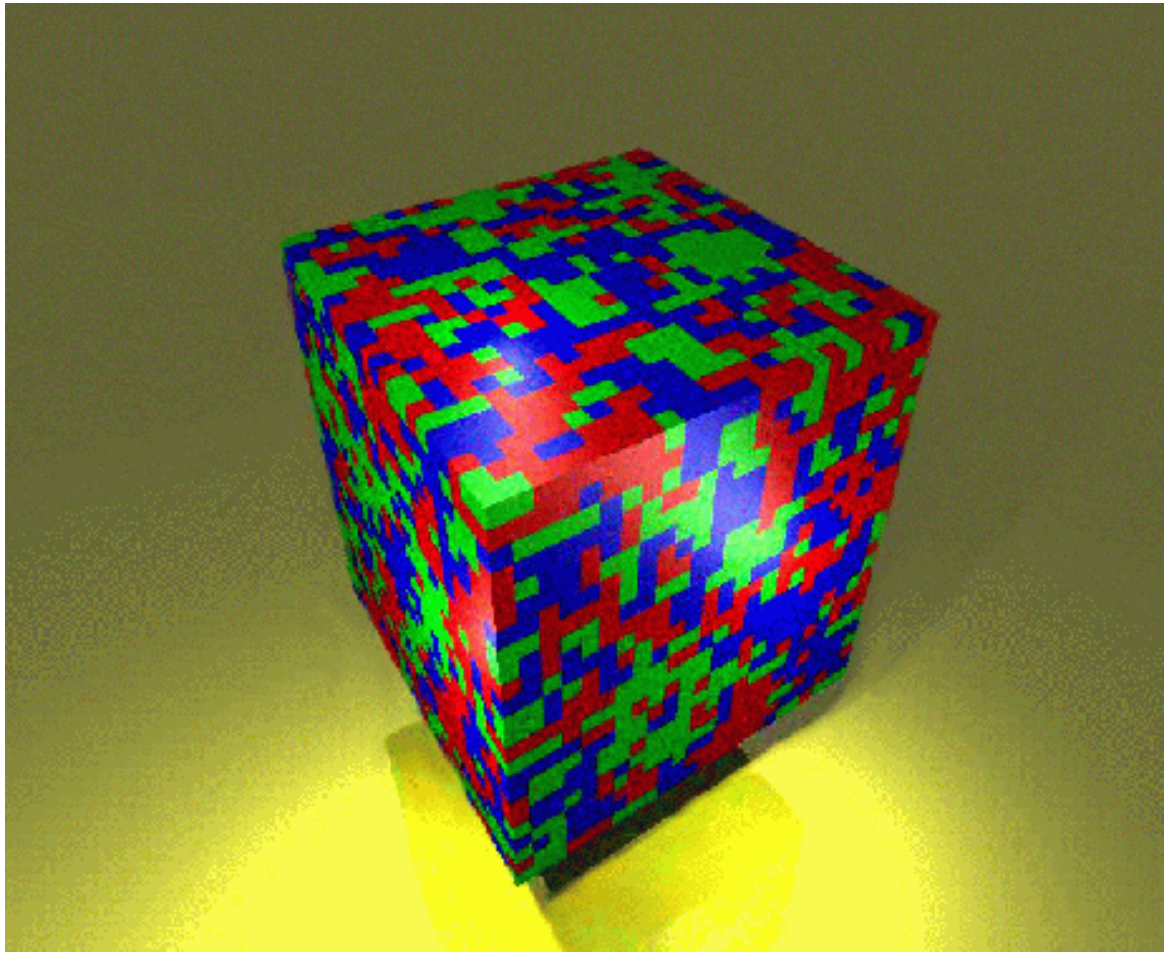


OUTLINE:

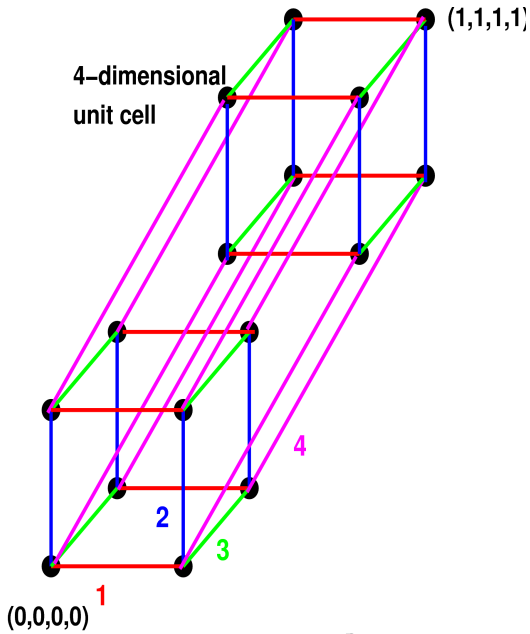
- lattice regularization and continuum limit
- QCD close to the chiral limit, O(N) scaling, phase diagram
- finite density QCD moments of charge fluctuations as probe for proximity to criticality

QCD thermodynamics at non-zero temperature and density

THERMODYNAMICS: $Z(V,T,\mu) = \mathrm{Tr}_V \mathrm{e}^{-\frac{1}{T}(\hat{H}-\mu\hat{N})}$

Euclidean path integral: $\tau \equiv it \quad \Rightarrow \quad \tau \in [0, 1/T)$

partition function: $Z(V,T,\mu) = \int \mathcal{D}\mathcal{A}\mathcal{D}\psi\mathcal{D}\bar{\psi}\ \mathrm{e}^{-S_E}$



$$\boxed{S_E = \int_0^{1/T} dx_0 \int_V d^3x\ \mathcal{L}_E(\mathcal{A},\psi,\bar{\psi},\mu)}$$

temperature volume chemical potential

QCD:

$$\mathcal{L}_E = \frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + \bar{\psi}_{j,a}\left(\sum_{\nu=0}^{3}\gamma_\nu\left(i\partial_\nu + \frac{g}{2}\mathcal{A}_\nu - i\mu\delta_{0,\nu}\right) - m_j\right)^{a,b}\psi_{j,b}$$

$a,b = 1,...,N_c^2-1$, colour

$j = 1,...,n_f$, flavour

Path Integrals and Thermodynamics in Quantum Mechanics

Consider a Hamiltonian, H, for a particle in a potential V(q):

$$H = \frac{p^2}{2m} + V(q) = -\frac{\hbar}{2m}\left(\frac{\mathrm{d}}{\mathrm{d}q}\right)^2 + V(q) = H_0 + V(q)$$

The Hamiltonian, H, will have a discrete Eigenvalue spectrum, if $V(q) \to \infty$ for $|q| \to \infty$.

$$H|\psi_n\rangle = E_n|\psi_n\rangle \ , \ E_n < E_{n+1} \ , \ n = 0,\ 1,\ \ldots$$

The time evolution operator U(t,t') describes the time-dependent states,

$$|\psi_n\rangle_t \equiv U(t,0)|\psi\rangle \ \text{ or } \ |\psi_n\rangle_t \equiv U(t,t')|\psi\rangle_{t'}$$

The time evolution operator obeys a Schroedinger equation:

$$\frac{\partial}{\partial t}U(t,t') = -\frac{i}{\hbar}HU(t,t') \ \Rightarrow \ U(t,t') = \exp\left(-\frac{i}{\hbar}(t-t')H\right)$$

Path Integrals in Quantum Mechanics, cont'd

Green's functions: $$G(y,t'';x,t') \equiv \langle y|U(t'',t')|x\rangle = \langle y|\mathrm{e}^{-i(t''-t')H/\hbar}|x\rangle$$

Time evolution of the state $|x\rangle_{t'}$ into the state $|y\rangle_{t''}$

Green's function of a free particle of mass m:

$$G(x,t;y,0) = \langle x|\mathrm{e}^{-iH_0t/\hbar}|y\rangle = \iint dp\,dp'\,\langle x|p\rangle\langle p|\mathrm{e}^{-iH_0t/\hbar}|p'\rangle\langle p'|y\rangle$$

Eigenstates of the momentum operator:

$$\delta(p-p')\mathrm{e}^{-\frac{i}{\hbar}\frac{p^2}{2m}t}$$

$$= \int dp\,\frac{1}{2\pi\hbar}\mathrm{e}^{ip(x-y)/\hbar}\mathrm{e}^{-\frac{i}{\hbar}\frac{p^2}{2m}t}$$

$$= \left(\frac{m}{2\pi\hbar it}\right)^{1/2}\mathrm{e}^{i\frac{m\hbar}{2t}(x-y)^2}$$

Path integral representation of the Green's function

As H commutes with itself, the time evolution operator can be split into an arbitrary number of (infinitesimal) time steps:

$$U(t'',t') = \prod_{m=1}^{n} U(t'+m\epsilon, t'+(m-1)\epsilon) \quad , \quad n\epsilon = t''-t'$$

lattice spacing !!

With this one can rewrite the Green's function, using eigenstates of the coordinate operator:

$$G(y,t'';x,t') = \langle y|U(t'',t')|x\rangle$$

$$= \int \prod_{m=0}^{n} dz_m \langle z_{m+1}|U(t'+m\epsilon, t'+(m-1)\epsilon)|z_m\rangle \quad ,$$

$$z_0 \equiv x \quad , \quad z_{n+1} \equiv y$$

Need a representation of this matrix element for small ϵ

$$\langle x|\mathrm{e}^{-\frac{i}{\hbar}\epsilon H}|y\rangle$$

Path integral representation of the Green's function

$$\langle x|\mathrm{e}^{-\frac{i}{\hbar}\epsilon H}|y\rangle : \quad H = H_0 + V(\hat{q})$$

$$\langle x|\mathrm{e}^{-\frac{i}{\hbar}\epsilon H}|y\rangle = \int dx' \langle x|\mathrm{e}^{-\frac{i}{\hbar}\epsilon H_0}|x'\rangle \langle x'|\mathrm{e}^{-\frac{i}{\hbar}\epsilon V(\hat{q})}|y\rangle \; (1 + \mathcal{O}(\epsilon^2))$$

use Baker-Hausdorff to split exponentials: $\mathrm{e}^A \mathrm{e}^B = \mathrm{e}^{A+B+[A,B]+\ldots}$

$$\begin{aligned}
\langle x|\mathrm{e}^{-\frac{i}{\hbar}\epsilon H}|y\rangle &= \langle x|\mathrm{e}^{-\frac{i}{\hbar}\epsilon H_0}|y\rangle \mathrm{e}^{-\frac{i}{\hbar}\epsilon V(y)} \\
&= \left(\frac{m}{2\pi\hbar i\epsilon}\right)^{1/2} \exp\left[\frac{i}{\hbar}\epsilon\left(\frac{m}{2}\left(\frac{y-x}{\epsilon}\right)^2 - V(x)\right)\right] \\
&\quad + \mathcal{O}(\epsilon^2)
\end{aligned}$$

discretization errors

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \frac{y-x}{\epsilon} + \mathcal{O}(\epsilon)$$

Path integral representation of the Green's function

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$$\langle x|\mathrm{e}^{-\frac{i}{\hbar}\epsilon H}|y\rangle = \int dx' \langle x|\mathrm{e}^{-\frac{i}{\hbar}\epsilon H_0}|x'\rangle \langle x'|\mathrm{e}^{-\frac{i}{\hbar}\epsilon V(\hat{q})}|y\rangle \; (1 + \mathcal{O}(\epsilon^2))$$

use Baker-Hausdorff to split exponentials: $\mathrm{e}^A \mathrm{e}^B = \mathrm{e}^{A+B+[A,B]+\ldots}$

$$\langle x|\mathrm{e}^{-\frac{i}{\hbar}\epsilon H}|y\rangle = \langle x|\mathrm{e}^{-\frac{i}{\hbar}\epsilon H_0}|y\rangle \mathrm{e}^{-\frac{i}{\hbar}\epsilon V(y)}$$

$$= \left(\frac{m}{2\pi\hbar i\epsilon}\right)^{1/2} \exp\left[\frac{i}{\hbar}\epsilon\left(\frac{m}{2}\left(\frac{y-x}{\epsilon}\right)^2 - V(x)\right)\right]$$

$$+\mathcal{O}(\epsilon^2)$$

discretization error == cut-off effects

Sign changed: from Hamiltonian to Lagrangian !!

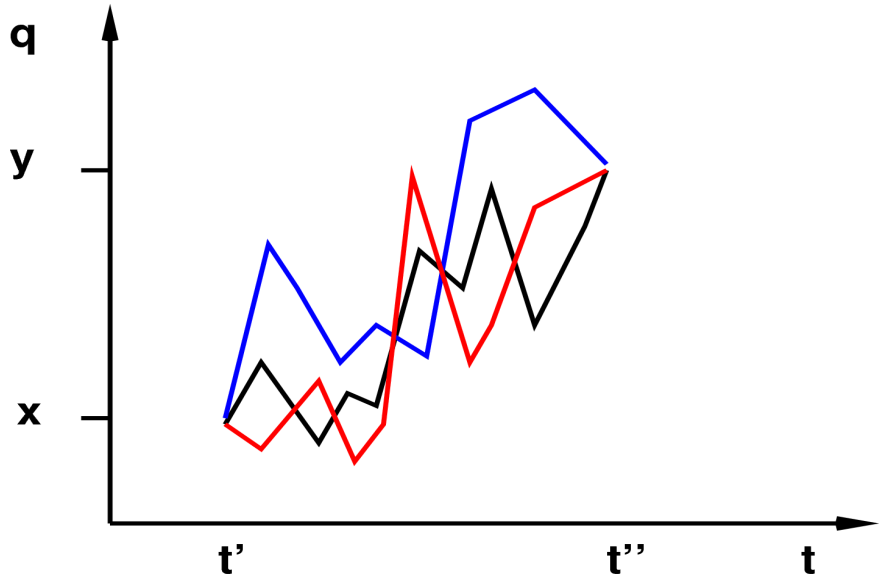
Path integral representation of the Green's function

$$G(y,t'';x,t') = \int_{q(t')=x}^{q(t'')=y} \mathcal{D}q(t)\ \mathrm{e}^{\frac{i}{\hbar}S(q)}$$

$$S(q) = \int_{t'}^{t''} dt \left(\frac{1}{2} m \left(\frac{\mathrm{d}q(t)}{\mathrm{d}t} \right)^2 - V(q(t)) \right)$$

classical action

L(q,q') :Lagrangian



weighted sum over all paths connecting x(t') and y(t'')

Euclidean formulation

rotate to imaginary time; $t \to -i\tau \quad , \quad \theta = \pi/2$

$$S_{\pi/2}(q) = i \int_0^\tau d|t| \left(\frac{m}{2} \left(\frac{\mathrm{d}q(t)}{\mathrm{d}|t|} \right)^2 + V(q(t)) \right)$$

$$\equiv iS_E(q)$$

Euclidean action: $S_E(q)$

partition function: $\tilde{Z}(-i\tau) \equiv Z(\tau) = \mathrm{Tr}\ \mathrm{e}^{-H\tau/\hbar} \quad , \quad t = -i\tau$

imaginary time

advantages:

- no oscillating terms; convergence properties of integrals under control

- probability interpretation $\mathrm{e}^{-S_E(q)} \geq 0 \quad , \quad \mathrm{e}^{-E_n\tau} \geq 0$

Euclidean formulation $\Longleftrightarrow$ Thermodynamics

$$Z_E(\tau) = \int dx \int_{q(0)=x}^{q(\tau)=x} \mathcal{D}q(\tau) \mathrm{e}^{-S_E(\tau)}$$

choose periodic paths, i.e. y=x

$$= \mathrm{Tr}\ \mathrm{e}^{-H\tau}$$

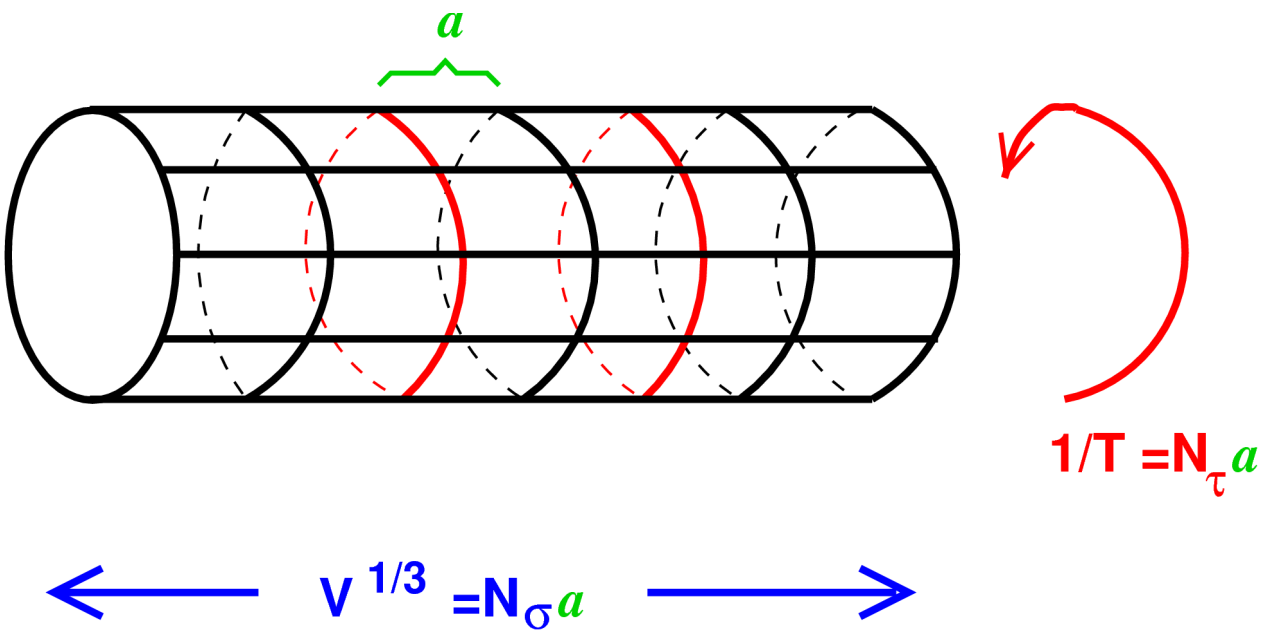
$$= \sum_n \mathrm{e}^{-E_n\tau} = \mathrm{e}^{-E_0\tau} \left(1 + \sum_{n>0} \mathrm{e}^{-(E_n-E_0)\tau}\right)$$

ground state energy: $E_0 = \lim_{\tau\to\infty} \frac{1}{\tau} \ln Z_E(\tau)$

Euclidean path integral over all paths with period τ
is the partition function of a system at temperature $T = \hbar/\tau$

$$Z(T) = \mathrm{Tr}\ \mathrm{e}^{-\beta H} = \sum_n \mathrm{e}^{-\beta E_n} \quad , \quad \beta \equiv 1/T$$

Simulating strongly interacting matter on a discrete space-time grid (lattice QCD)



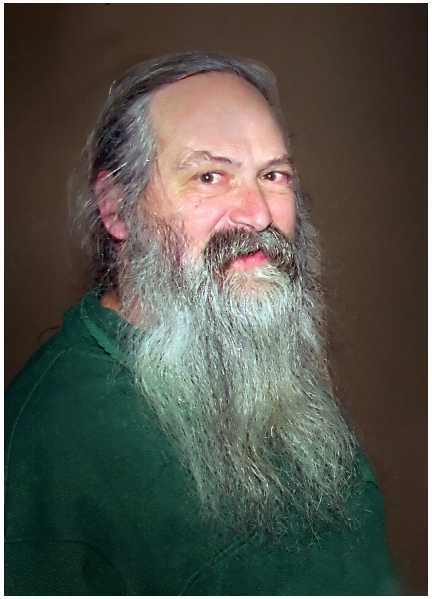
the lattice: $N_\sigma^3 \times N_\tau$

lattice spacing: a

partition function:

$$Z(V,T,\mu) = \int \mathcal{D}\mathcal{A}\mathcal{D}\psi\mathcal{D}\bar{\psi}\; \mathrm{e}^{-S_E}$$

$$S_E = \int_0^{1/T} dx_0 \int_V d^3x\; \mathcal{L}_E(\mathcal{A},\psi,\bar{\psi},\mu)$$



Mike Creutz

T: temperature V: volume μ : chemical potential

Discretizing Quantumchromodynamics

Euclidean action of QCD: $S_E(A_\mu, \bar{\psi}, \psi) \equiv S_G(A_\mu) + S_F(A_\mu, \bar{\psi}, \psi)$

Fermions: $S_F = \int d^4x \bar{\psi}(x) \left(\gamma_\mu D_\mu(A_\mu) + m\right) \psi(x)$

4-spinor:
$\psi \equiv (\psi_1, ..., \psi_4)$

covariant derivative: $D_\mu \equiv \partial_\mu + igA_\mu(x)$

γ-matrices anti-commute: $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu} \cdot 1$

Gauge fields: $S_G = \frac{1}{2} \int \mathrm{d}^4x \, \mathrm{Tr} F_{\mu\nu}(x) F_{\mu\nu}(x)$

non-abelian field strength tensor

$$F_{\mu\nu} = \frac{1}{ig}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$$

$$F^\alpha_{\mu\nu} = \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu - \frac{g}{2} f^{\alpha\beta\gamma} A^\beta_\mu A^\gamma_\nu \quad , \quad F_{\mu\nu} = \sum_{\alpha=1}^{N_c^2-1} F^\alpha_{\mu\nu} \frac{\lambda^\alpha}{2}$$

THE IDEA

lattice discretization:

I) introduce gauge degrees of freedom on links;
II) choose compact link-variables

$$U_\mu(n) \in SU(N_c) \quad , \quad U_\mu(n) \equiv e^{iagA_\mu(n)}$$

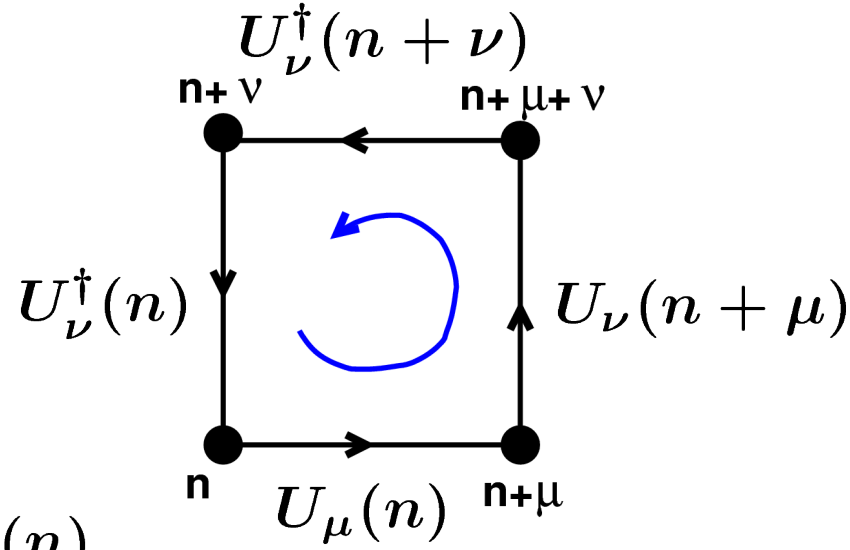
$N_c \times N_c$ matrix

$$A_\mu = \sum_{\alpha=1}^{N_c^2-1} A_\mu^\alpha \frac{\lambda^\alpha}{2}$$

K. Wilson,
Phys. Rev. D 10 (1974) 2445

the plaquette variable $\mathrm{Re\ Tr}\ U_{\mu\nu}(n)$

gauge invariance
$U' = G_x U G_y^\dagger$



$$U_{\mu\nu}(n) \equiv U_\mu(n) U_\nu(n+\mu) U_\mu^\dagger(n+\nu) U_\nu^\dagger(n)$$

naïve continuum limit

$$U_{\mu\nu}(n) \equiv U_\mu(n) U_\nu(n+\mu) U_\mu^\dagger(n+\mu) U_\nu^\dagger(n)$$

$$= \underbrace{\mathrm{e}^{iagA_\mu(n)} \mathrm{e}^{iagA_\nu(n+\mu)}}_{\mathrm{e}^{iag(A_\mu(n)+A_\nu(n+\mu)) - a^2 g^2 [A_\mu(n), A_\nu(n+\mu)]}} \mathrm{e}^{-iagA_\mu(n+\mu)} \mathrm{e}^{-iagA_\nu(n)}$$

use Baker-Hausdorff repeatedly

$$\Rightarrow \quad \mathrm{Re\ Tr} U_{\mu\nu}(n) = N_c - a^4 \frac{g^2}{2N_c} \mathrm{Tr}\ F_{\mu\nu}(n) F_{\mu\nu}(n) + \mathcal{O}(a^6)$$

$$S_G = \frac{2N_c}{g^2} \sum_n \sum_{1 \le \mu < \nu \le 4} \left(1 - \frac{1}{N_c} \mathrm{Re\ Tr}\ U_{\mu\nu}(n) \right)$$

discretization error $\mathcal{O}(a^2)$

$$Z(\beta) = \int \prod_{n,\mu} dU_\mu(n) \mathrm{e}^{-\beta S_G(U)} \quad , \quad \beta \equiv \frac{2N_c}{g^2(a)}$$

better: $a(g^2)$

Discretization of the fermion sector

Fermions: $$S_F = \int d^4x \bar{\psi}(x) \left(\gamma_\mu(\partial_\mu + igA_\mu(x)) + m\right) \psi(x)$$

discretization of first order derivative in fermionic part is straightforward:

$$\bar{\psi}(x)\partial_\mu\psi(x) \quad \rightarrow \quad \bar{\psi}_n \left(\frac{\psi_{n+\mu} - \psi_n}{a}\right)$$

$$\bar{\psi}_n\psi_{n+\mu} - \bar{\psi}_n\psi_n$$

discretization of derivative generates point-split terms $\Rightarrow$ local gauge invariance?

improved actions: higher order difference schemes allow to reduce discretization errors

Discretization of the fermion sector

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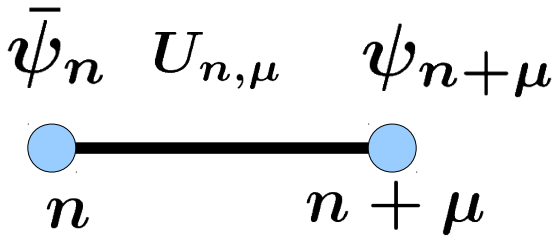
$$U_{n,\mu} = \mathrm{e}^{iagA_\mu(x)}$$

takes care of gauge invariance and generates covariant derivative



Improved discretization of the fermion sector

staggered fermions: $S_F = \sum_{ij} \bar{\chi}_i M_{ij} \chi_j$

$\gamma_\mu \Rightarrow$ staggered phase

$\eta_{x,\mu} = (-1)^{x_1 + x_2 \ldots + x_{\mu-1}}$

discretization of first order derivative in fermionic part:

$$M[U]_{ij} = m\delta_{i,j} + \left(\frac{9}{16} A\,[U]_{i\,j} - \frac{1}{48} B\,[U]_{i\,j} \right)$$

naive, 1-link

$\mathcal{O}(a^2)$ improved, 3-link

$$A[U]_{ij} = \sum_\mu \eta_{i,\mu} \left(U_{i,\mu} \delta_{i,j-\hat{\mu}} - U^\dagger_{i-\hat{\mu},\mu} \delta_{i,j+\hat{\mu}} \right)$$

$$B[U]_{ij} = \sum_\mu \eta_{i,\mu} \left(U_{i,\mu} U_{i+\hat{\mu},\mu} U_{i+2\hat{\mu},\mu} \delta_{i,j-3\hat{\mu}} - U^\dagger_{i-\hat{\mu},\mu} U^\dagger_{i-2\hat{\mu},\mu} U^\dagger_{i-3\hat{\mu},\mu} \delta_{i,j+\hat{\mu}} \right)$$

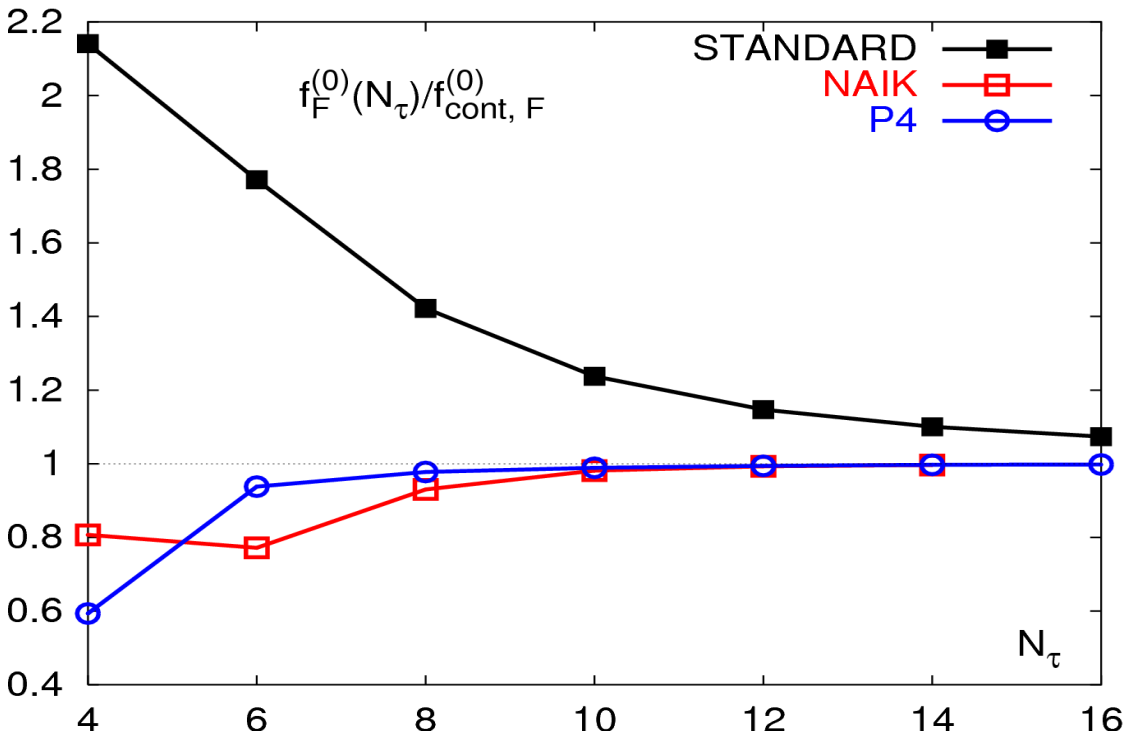
$\Rightarrow$

1-link: standard staggered, stout,...

3-link: Naik, asqtad, HISQ,... $\Leftarrow$ drastically improved high-T behavior

Cut-off effects in the infinite temperature limit

QCD, $T \to \infty$: asymptotic freedom, ideal gas



free energy: $\dfrac{f_F(N_\tau)}{f_{F,cont}} = 1 + \mathcal{O}(N_\tau^{-2})$ standard

$1 + \mathcal{O}(N_\tau^{-4})$ Naik, p4

Fermion Determinant

partition function again:

$$Z(V,T,\mu) = \int \mathcal{D}U \mathcal{D}\psi \mathcal{D}\bar{\psi}\ \mathrm{e}^{\bar{\psi} M(U,m_q,\mu)\psi}\ \mathrm{e}^{-S_G}$$

$$= \int \mathcal{D}\mathcal{U}\ \mathrm{Det} M(U,m_q,\mu)\ \mathrm{e}^{-S_G}$$

The fermion determinant

– encodes the symmetries of the fermion sector of QCD

– is not positive definite for non-zero chemical potential

other discretization schemes:

- Wilson fermions (clover fermions)
- domain wall fermions
- overlap fermions

The continuum limit of lattice regularized QCD

- at present we consider only SU(N) gauge theories

- the only free parameter is the gauge coupling g or $\beta = 2N_c/g^2$

$$\Rightarrow \quad g^2 \equiv g^2(a)$$

Continuum limit: $a \longrightarrow 0$

Continuum limit: physical observables, $\Gamma = m_G,\ \sigma,\ r_0,\ \epsilon,\ \ldots$
do not depend on the lattice spacing (cut-off)

$$a\frac{\mathrm{d}}{\mathrm{d}a}\Gamma = \left(a\frac{\partial}{\partial a} - \beta(g)\frac{\partial}{\partial g}\right)\Gamma = 0$$

renormalization group equation

with $\beta(g) = -a\dfrac{\mathrm{d}g}{\mathrm{d}a}$ β -function

Asymptotic Freedom

$$\Rightarrow \quad \beta(g) = -\frac{11}{16\pi^2}\, g^3 + \mathcal{O}(g^5) \qquad (N_c = 3,\ n_f = 0)$$

$$a\Lambda_L \quad = \quad \mathrm{e}^{-\frac{1}{2b_0}\frac{1}{g^2}} \qquad \Leftrightarrow \quad g^2(a) = \frac{1}{2b_0 \ln(1/a\Lambda_L)}$$

in massless QCD Λ_L is the only 'free parameter' that sets the scale for ALL PHYSICAL OBSERVABLES.

related to Λ -parameter in other regularization schemes, e.g.: $\Lambda_L = c_L \Lambda_{\overline{MS}}$

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the next order: $\beta(g) = -b_0 g^3 - b_1 g^5 + \mathcal{O}(g^7)$

$$a\Lambda_L \quad = \quad R(g) = \left(b_0 g^2\right)^{-b_1/2b_0^2}\ \mathrm{e}^{-\frac{1}{2b_0}\frac{1}{g^2}}$$

re-introducing color factors and non-zero flavor numbers

$$b_0 \quad = \quad \frac{1}{16\pi^2}\left(11\frac{N_c}{3} - \frac{2}{3}n_f\right)$$

$$b_1 \quad = \quad \frac{1}{(16\pi^2)^2}\left(\frac{34}{3}N_c^2 - \left(\frac{10}{3}N_c + \frac{N_c^2-1}{N_c}\right)n_f\right)$$

Asymptotic Freedom at work

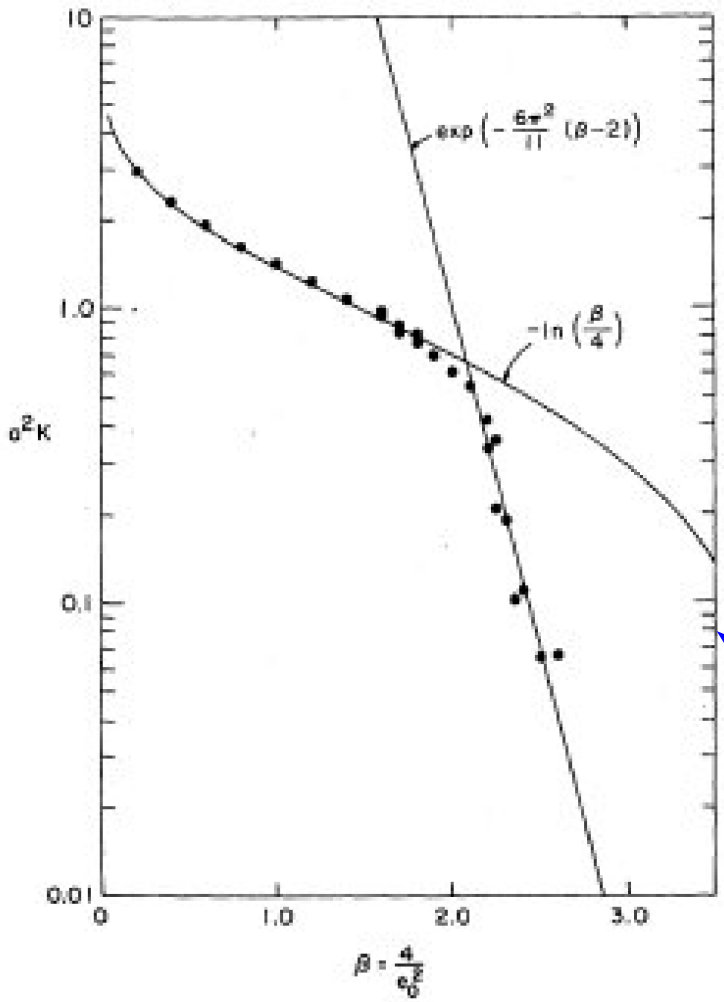
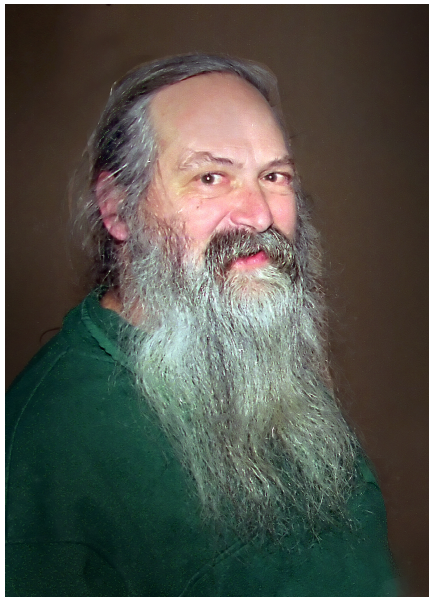


FIG. 6. The cutoff squared times the string tension
s a function of β. The solid lines are the strong- and
eak-coupling limits.

ow $\beta = 2.1$ only loops of side 1 and 2 are signifi-
antly different from zero so we must include the
oop of side 0 in the fit. Below $\beta = 1.6$ only the

Monte Carlo simulation 1979

Mike Creutz



non-zero string tension

confinement

universal weak coupling scaling behavior;
cut-off dependence is eliminated

$$a\Lambda_L = \left(b_0 g^2\right)^{-b_1/2b_0^2} \mathrm{e}^{-\frac{1}{2b_0}\frac{1}{g^2}}$$

$\Rightarrow \sigma/\Lambda^2$

Lines of constant physics

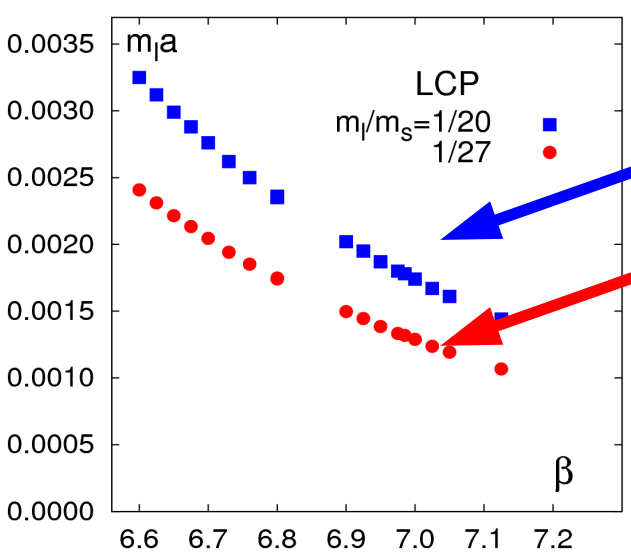
approach to continuum limit in QCD with several quark masses:

quark masses (unrenormalized, in lattice units): $m_u a = m_d a;\ m_s a;\ (m_c a)$

$$a \to 0 \ \Leftrightarrow\ \beta = 6/g^2 \to \infty$$

e.g. we need to control 3 parameters: $\Lambda_L,\ m_{u(d)},\ m_s$

- ★ for given β tune $m_l a \equiv m_{u(d)} a,\ m_s a$ such that 2 experimentally known observables, e.g. 2 hadron masses, are reproduced
- ★ need one additional observable to fix the physical scale; e.g. $f_K,\ r_0, \ldots$ or $\ldots m_H, \ldots.$



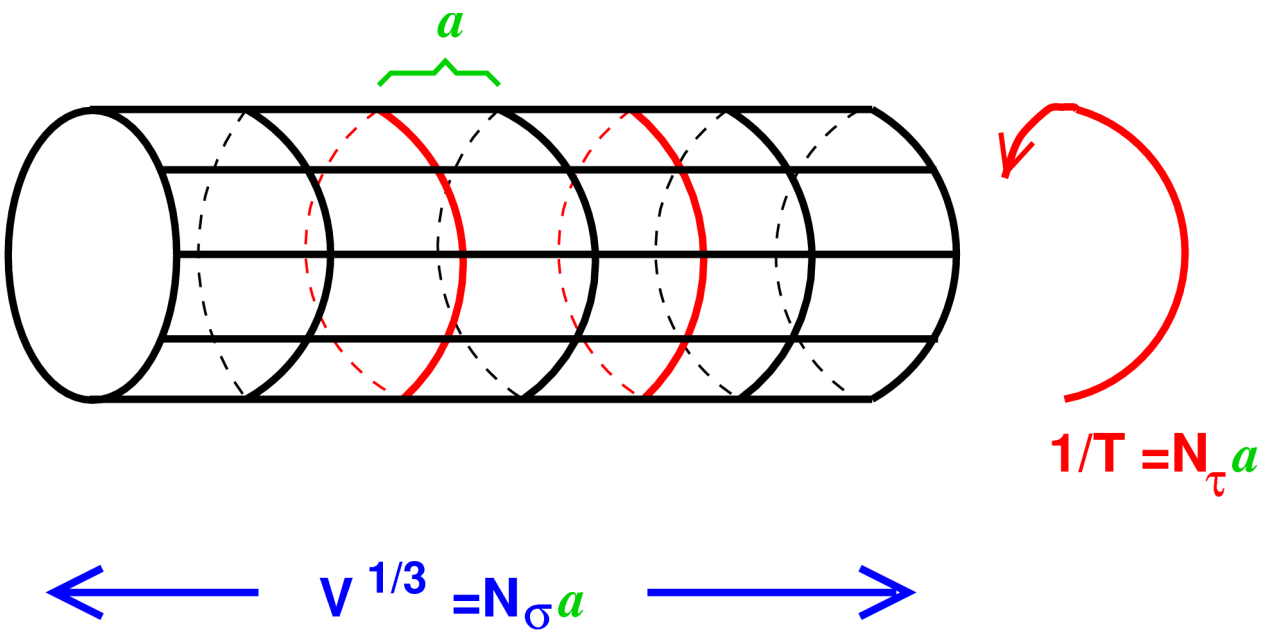
$m_\pi = 160$ MeV $\quad m_K = 495$ MeV

$m_\pi = 140$ MeV $\quad f_K = 156$ MeV

all parameters are now fixed for ALL T=0 as well as T>0 calculations

Ready to go.....

Simulating strongly interacting matter on a discrete space-time grid (lattice QCD)



the lattice: $N_\sigma^3 \times N_\tau$

lattice spacing: a

$$\frac{p}{T^4} = \ln Z$$

$$\frac{\epsilon}{T^4} = -\frac{1}{VT^4}\frac{\partial \ln Z}{\partial T^{-1}}$$

$$\frac{n_q}{T^3} = \frac{1}{VT^3}\frac{\partial \ln Z}{\partial \mu_q/T}$$

$$\frac{\chi_q}{T^2} = \frac{1}{VT^3}\frac{\partial^2 \ln Z}{\partial (\mu_q/T)^2}$$

partition function:

$$Z(V,T,\mu) = \int \mathcal{D}\mathcal{A}\mathcal{D}\psi\mathcal{D}\bar{\psi}\; \mathrm{e}^{-S_E}$$

$$S_E = \int_0^{1/T} dx_0 \int_V d^3x\; \mathcal{L}_E(\mathcal{A},\psi,\bar{\psi},\mu)$$

T: temperature V: volume μ : chemical potential

QCD equation of state

– calculating the equation of state on lines of constant physics

– the trace anomaly

$$\frac{\epsilon - 3p}{T^4} = T\frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{p}{T^4}\right) = \left(a\frac{\mathrm{d}\beta}{\mathrm{d}a}\right)_{LCP}\frac{\partial p/T^4}{\partial\beta}$$

$$= \left(\frac{\epsilon - 3p}{T^4}\right)_{gluon} + \left(\frac{\epsilon - 3p}{T^4}\right)_{fermion}$$

– the pressure: integrating the trace anomaly

$$\left.\frac{p}{T^4}\right|_{\beta_0}^{\beta} = \int_{T_0}^{T}\mathrm{d}T\,\frac{1}{T}\left(\frac{\epsilon - 3p}{T^4}\right)$$

– need T-scale and its relation to the gauge coupling: $a(\beta)$

$aT = 1/N_\tau$

Determination of the scale (lattice spacing)

– calculate an experimentally known observable, e.g. m_H

– a lattice calculation provides this observable in units of "a"

$$\hat{m}_H = m_H a \;\Rightarrow\; a = \frac{\hat{m}_H}{m_H}$$

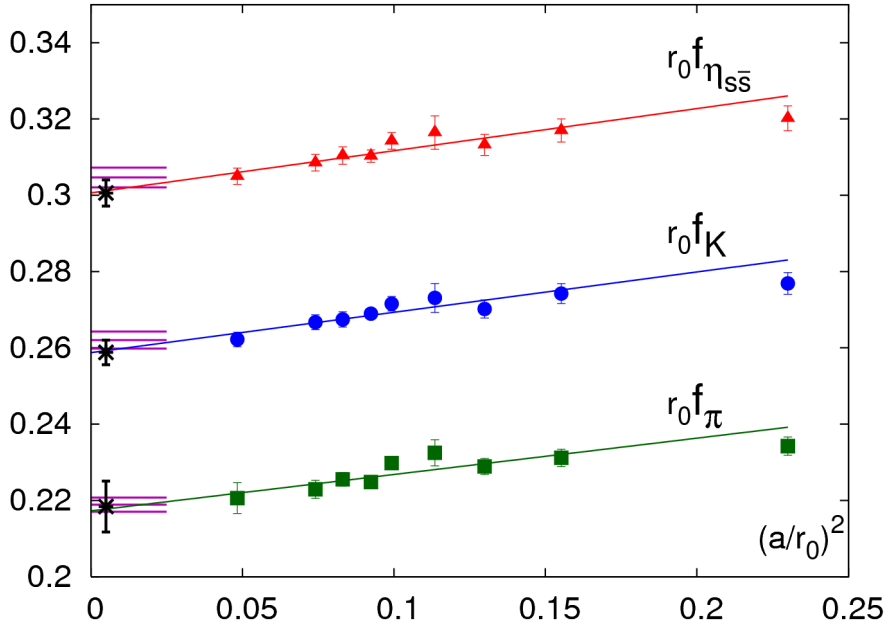
this "defines" the lattice spacing based on the calculation of a physical observable

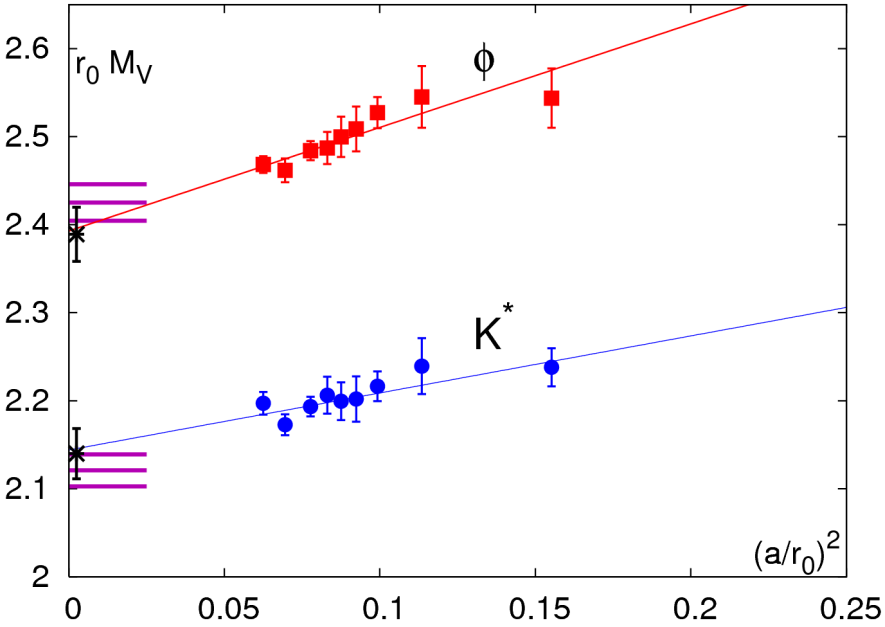
only in the continuum limit will all definitions of a lattice spacing based on different observables agree

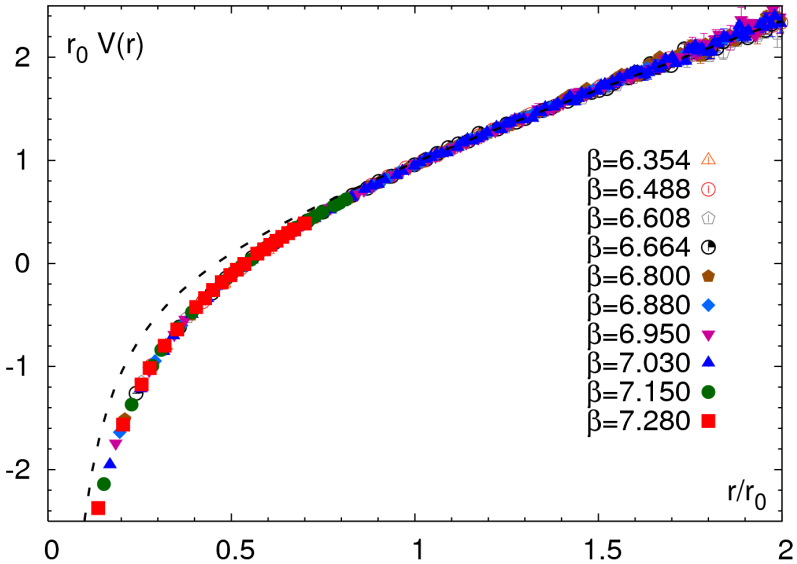
– temperature scale:

$$\hat{T} \equiv aT = 1/N_\tau \;\Rightarrow\; \frac{T}{m_H} = \frac{\hat{T}}{\hat{m}_H} = \frac{1}{m_H a N_\tau}$$

Several T=0 observables used to set a scale





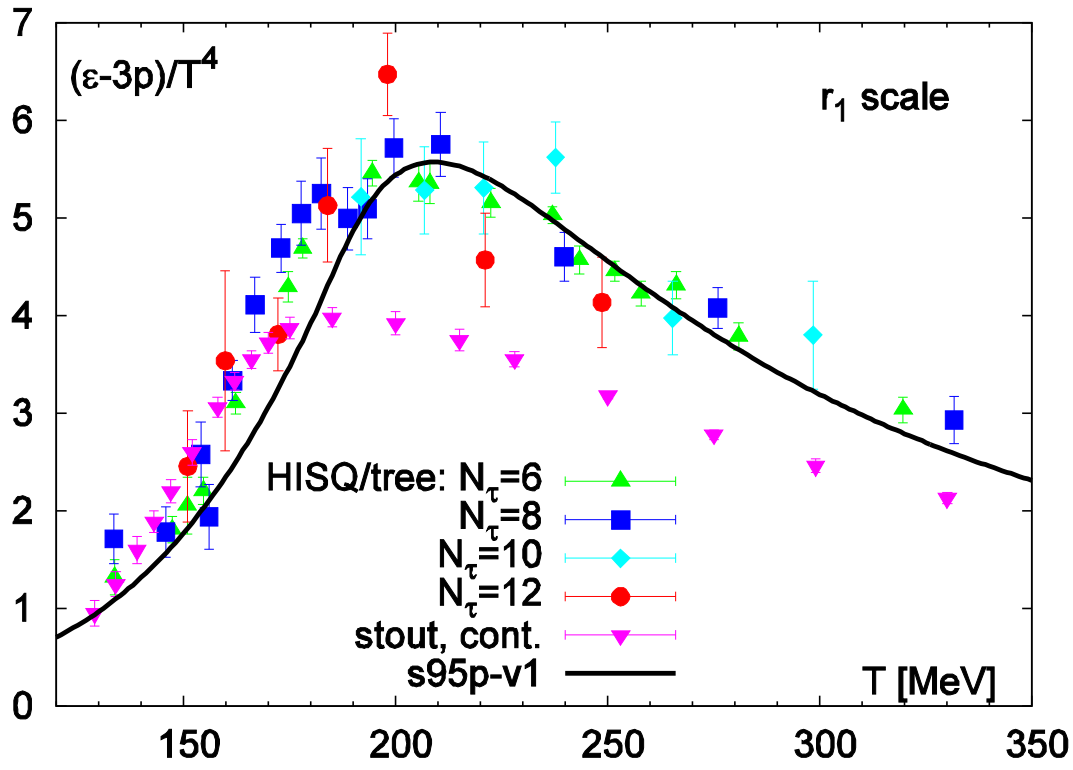


decay constants, hadron masses, heavy quark potential,.....

$$\left(r^2 \frac{\mathrm{d}V_{\bar{q}q}(r)}{\mathrm{d}r} \right)_{r=r_0} = 1.65$$

$$r_0 = 0.462(11)(4)\ \mathrm{fm}$$

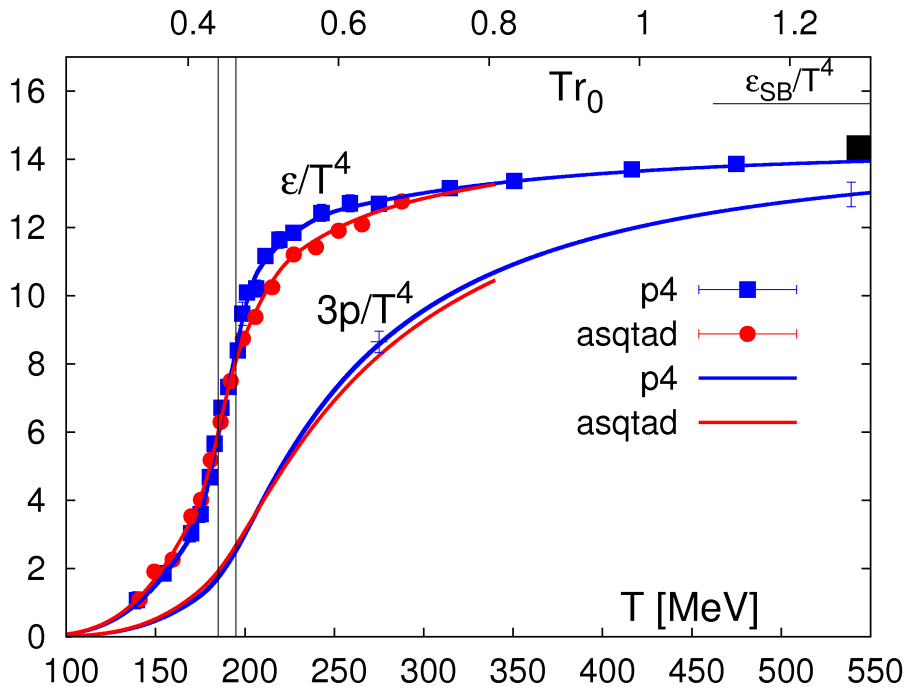
The trace anomaly: $(\epsilon - 3p)/T^4$



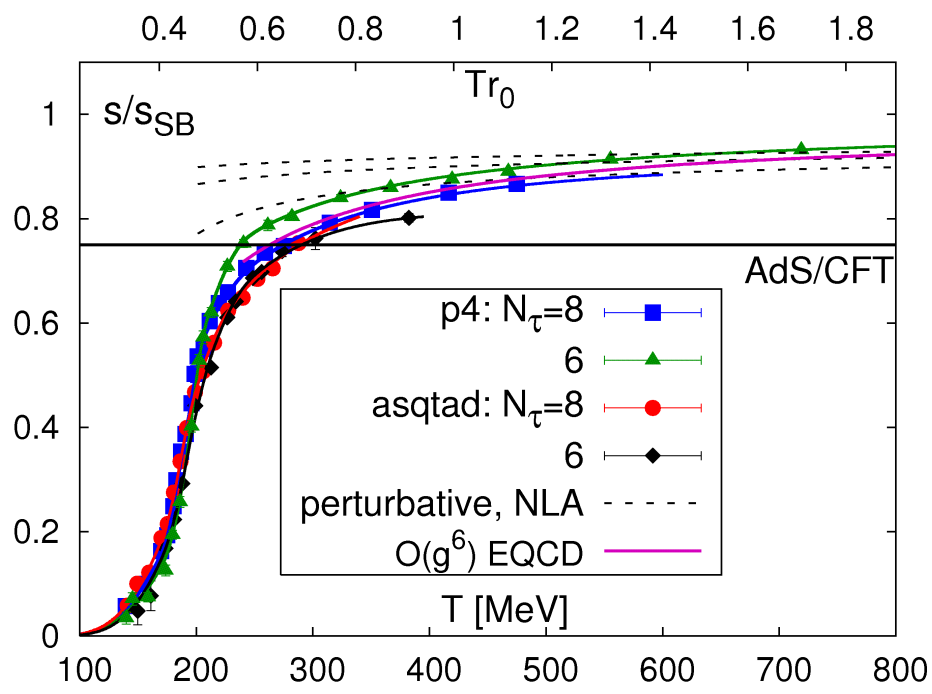
HISQ/tree: hotQCD preliminary

stout.cont.: estimate of continuum result based on average of Nt=8,10 data, Budapest-Wuppertal,1007.2580

Energy density, pressure, entropy density



Bazavov et al (HotQCD), PRD 80 (09) 14504



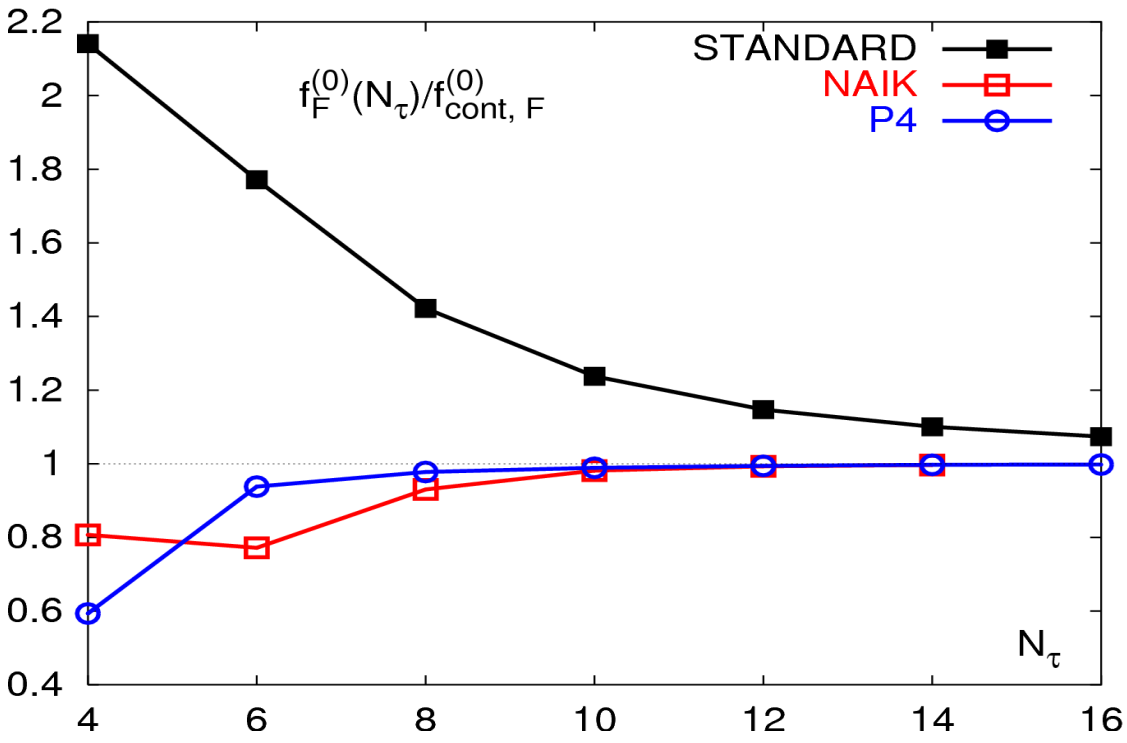
Petreczky, NPA 830 (10) 11c

rapid rise of the energy density reflects the "liberation" of many new degrees of freedom (?)

– deconfinement
– chiral symmetry restoration

Cut-off effects in the infinite temperature limit

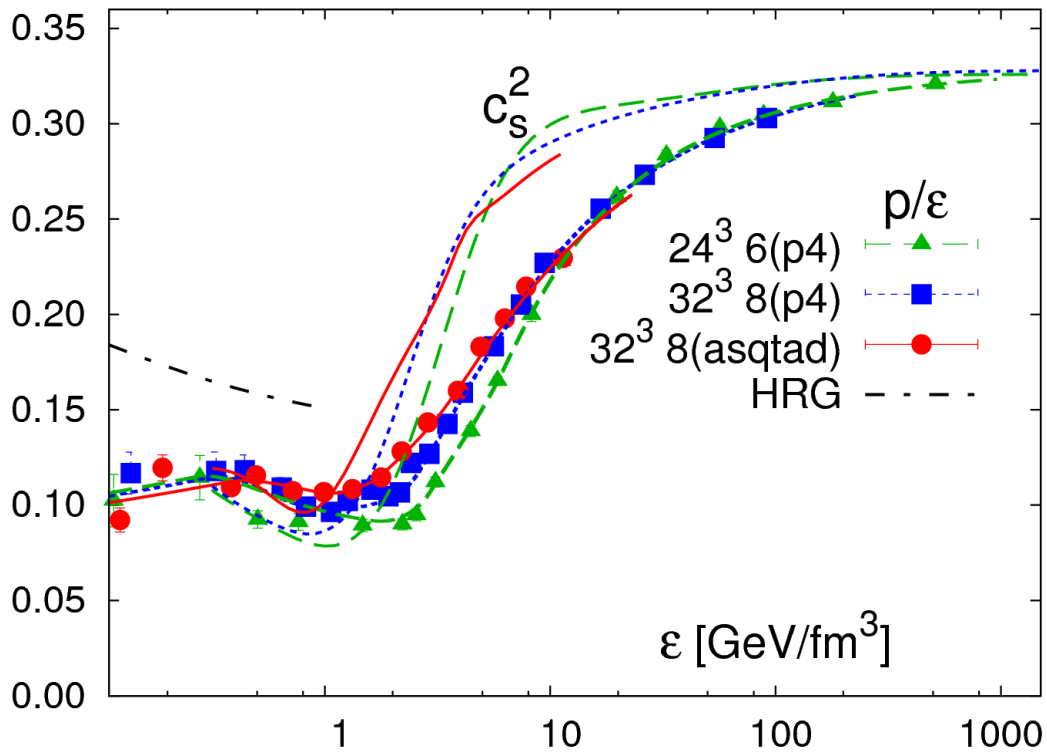
QCD, $T \to \infty$: asymptotic freedom, ideal gas



free energy: $\dfrac{f_F(N_\tau)}{f_{F,cont}} = 1 + \mathcal{O}(N_\tau^{-2})$ standard

$1 + \mathcal{O}(N_\tau^{-4})$ Naik, p4

Velocity of sound



A. Bazavov et al (hotQCD Coll.),
Phys. Rev. D80 (2009) 014504

The confinement – deconfinement transition

confinement

– stick together, find a comfortable distance
– controlled by the "confinement potential"

$$V_{\bar{q}q}(r) = -\frac{4}{3}\frac{\alpha(r)}{r} + \sigma\, r$$

deconfinement

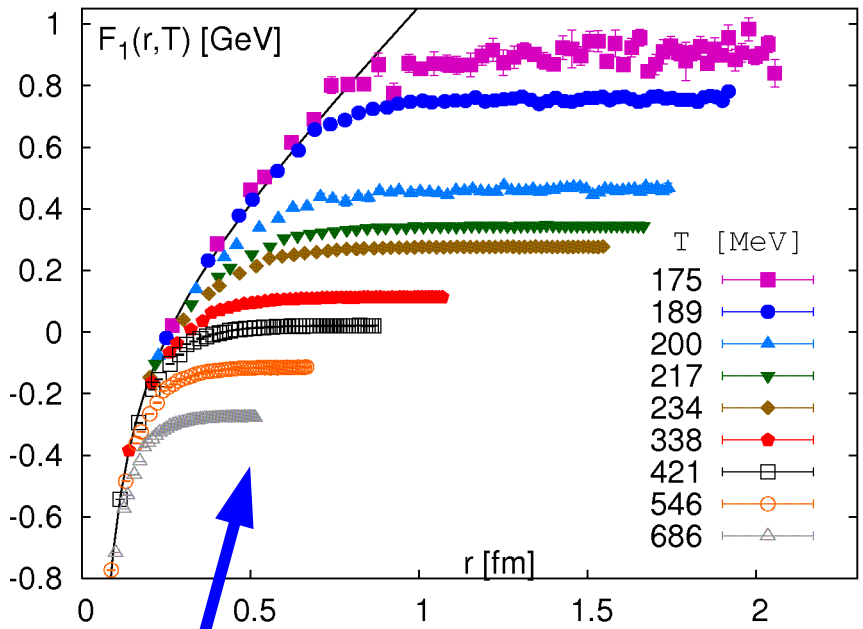
– freely floating in the crowed
– do not care what color your neighbor has

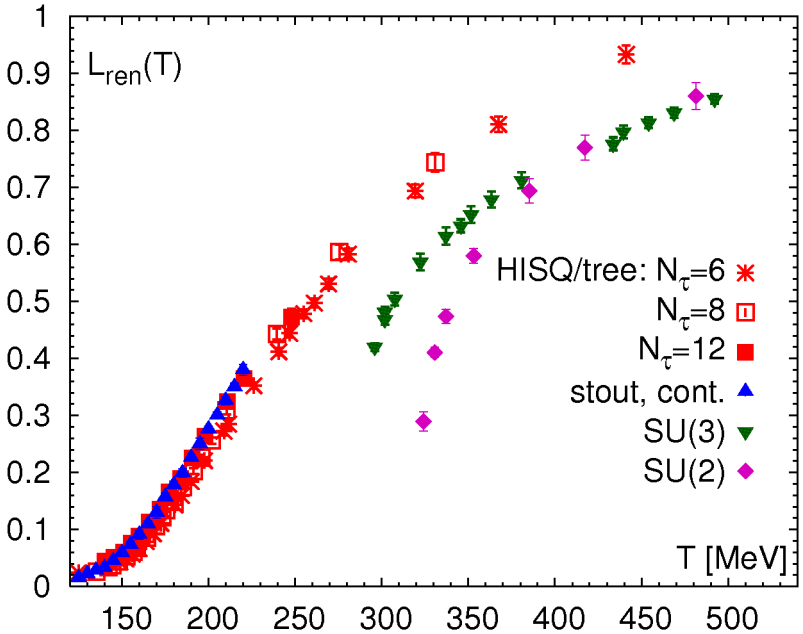
screening, color is neutralized on the average over a (short) distance

This transition happens abruptly: PHASE TRANSITION

Screening of static quark anti-quark pairs

$$L_{\vec{x}} = \mathrm{tr}\mathcal{P}\mathrm{e}^{ig\int_0^{1/T} d\tau A_0(\tau,\vec{x})}$$





free energy: $\mathrm{e}^{-F_{q\bar{q}}(r,T)/T} = Z_q^2 \langle \mathrm{Tr} L_0 \mathrm{Tr} L^\dagger_{|\vec{x}|} \rangle \quad , \quad r = |\vec{x}|$

Polyakov loop: $L^2_{\mathrm{ren}} \equiv \mathrm{e}^{-2\ F_q(T)/T} = \lim_{r\to\infty} Z_q^2 \langle \mathrm{Tr} L_0 \mathrm{Tr} L^\dagger_{|\vec{x}|} \rangle$

$$F_1(r,T) \simeq \frac{4}{3}\frac{\alpha}{r}\mathrm{e}^{-m_D\ r} + 2F_q(T)$$

Debye screened Coulomb potential

→ quarkonium melting