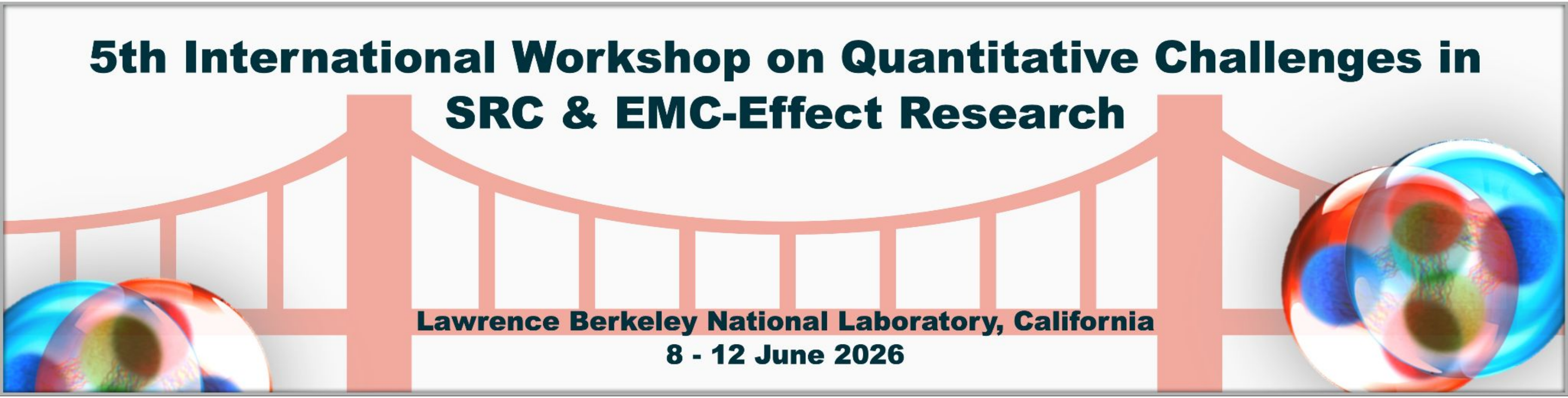
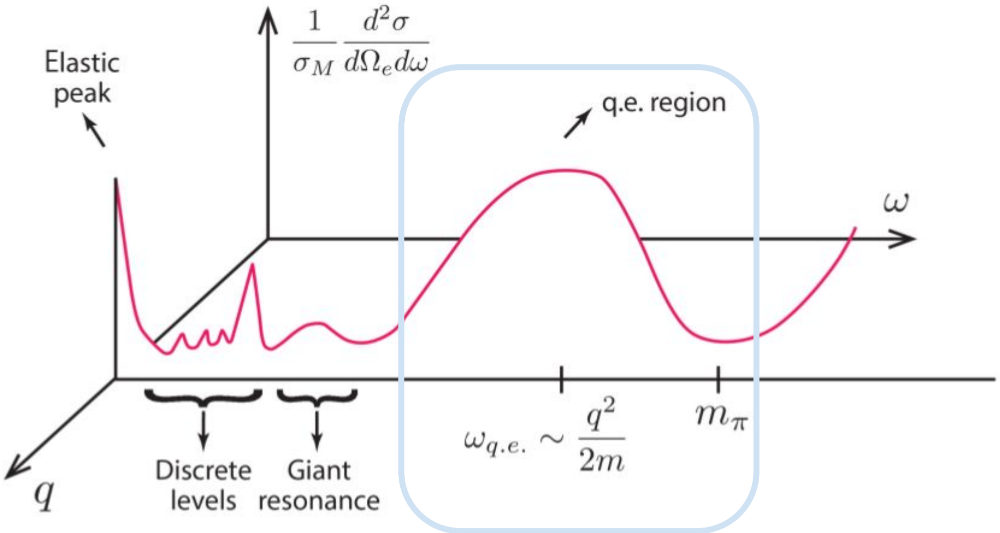
Electron Scattering from Light Nuclei with QMC methods

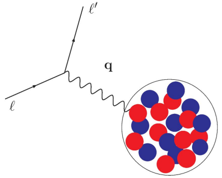
11 June 2026

Saori Pastore



Electron-Nucleus Scattering Cross Section

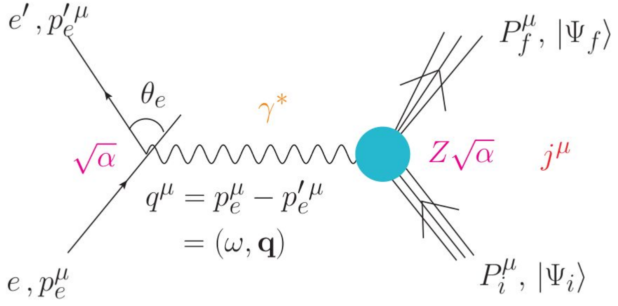


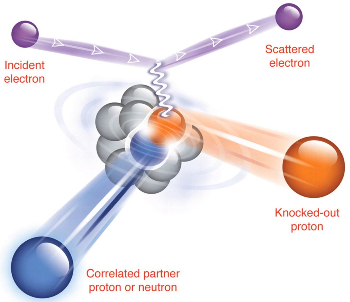


Energy and momentum transferred (ω,q)

Current and planned experimental programs rely on theoretical calculations at different kinematics

Electromagnetic probes





$$\lambda = \frac{h}{p}$$

α ~ 1/137 allows for a perturbative treatment of the EM.

x-sections are factorized into a well-known part specified by the electron kinematics and a part proportional to the matrix element of the EM current.

$$|\langle\Psi_f| j^\mu |\Psi_i\rangle|^2$$

The experimental data are (in most cases) known with great accuracy providing stringent constraints on theories.

For light nuclei, the many-body problem can be solved exactly of within controlled approximations.

EM observables used to validate the microscopic model.

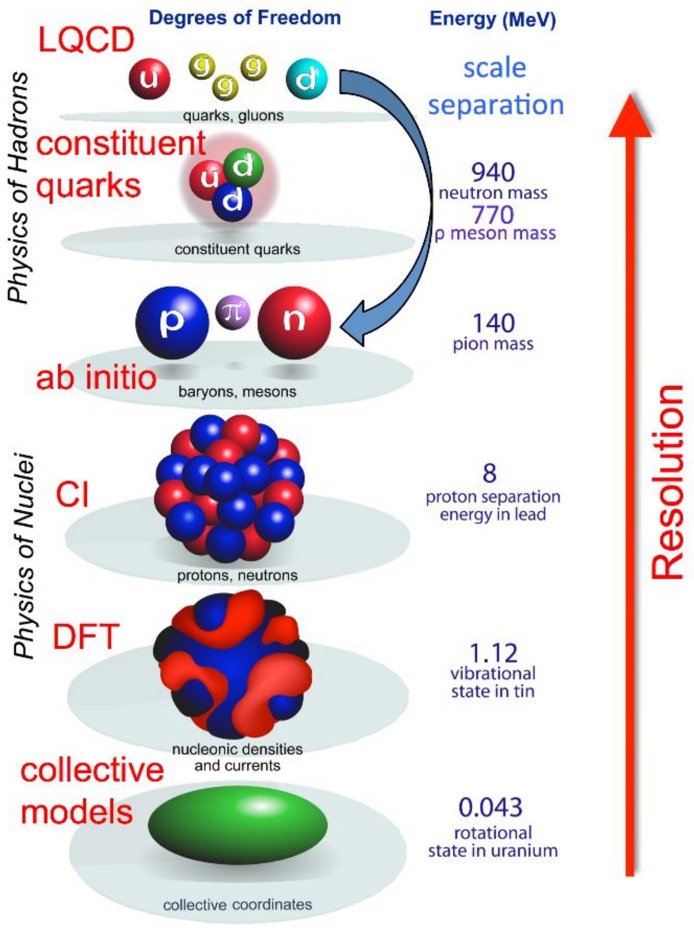
From Quarks to Nuclei

- Nuclei are complex systems made of interacting protons and neutrons, which in turns are composite objects made of interacting constituent quarks

- All fundamental forces are at play in nuclei

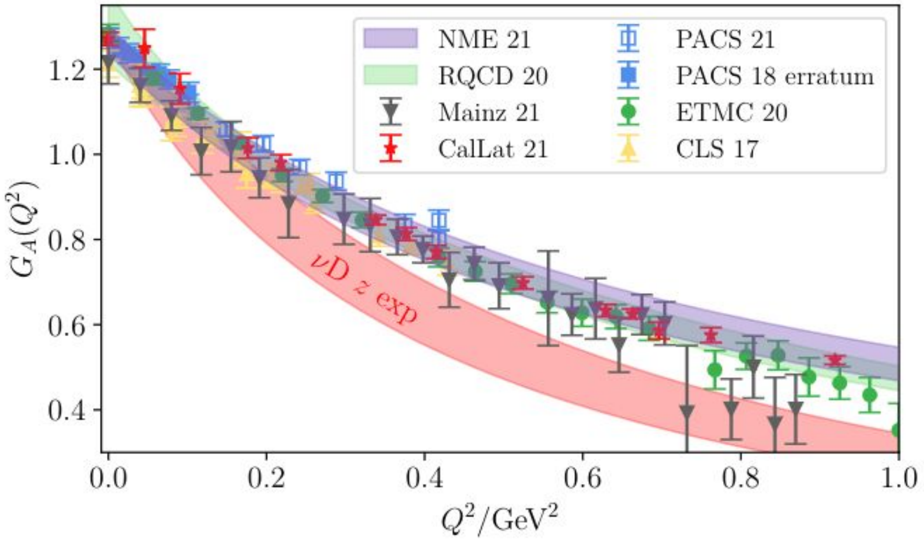
- **EFTs** low-energy approximations of QCD whose d.o.f. are bound states of QCD (e.g., protons, neutrons, pions, …); used to construct many-nucleon interactions and currents

- Accurate inputs at the single- and few-nucleon level are required (e.g., from **LQCD**)



LQCD for single- and few-nucleon properties

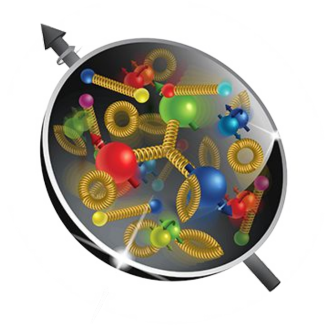
Microscopic approaches rely on accurate inputs at the single- and few-nucleon level from experimental data (where available) and Lattice QCD theoretical calculations.



Snowmass WP: Theoretical tools for neutrino scattering: interplay between lattice QCD, EFTs, nuclear physics, phenomenology, and neutrino event generators; arXiv:2203.09030, Meyer, Walker-Loud, Wilkinson (2022)

Building blocks of ab initio nuclear approaches:

Nucleonic form factors
Transition form factors
Pion production amplitudes
Two-nucleon couplings (strong and EW)
…



Microscopic (or *ab initio*) Description of Nuclei

Comprehensive theory that describes quantitatively and predictably nuclear structure and reactions

Requirements:

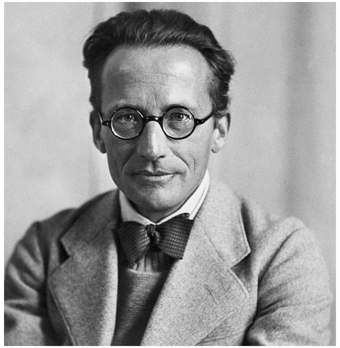
- Accurate understanding of the interactions/correlations between nucleons in **paris**, **triplets**, ... (**two- and three-nucleon forces**)

$$H = \sum_i -\frac{\hbar^2}{2m}\nabla_i^2 + \sum_{i<j} v_{ij} + \sum_{i<j<k} V_{ijk}$$

- Accurate understanding of the electroweak interactions of external probes (electrons, neutrinos, photons) with nucleons, correlated nucleon-pairs, … (**one- and two-body electroweak currents**)

$$O_\alpha(\mathbf{q}) = \sum_i O_i^{(\alpha)}(\mathbf{q}) + \sum_{i<j} O_{ij}^{(\alpha)}(\mathbf{q}) + \cdots$$

- **Computational methods** to solve the many-body nuclear problem of strongly interacting particles



Erwin Schrödinger

$$H\Psi = E\Psi$$

Many-body Nuclear Problem

Nuclear Many-body Hamiltonian

$$H = T + V = \sum_{i=1}^{A} t_i + \sum_{i<j} v_{ij} + \sum_{i<j<k} V_{ijk} + \dots$$

http://exascaleage.org/np/

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_A, s_1, s_2, \dots, s_A, t_1, t_2, \dots, t_A)$$

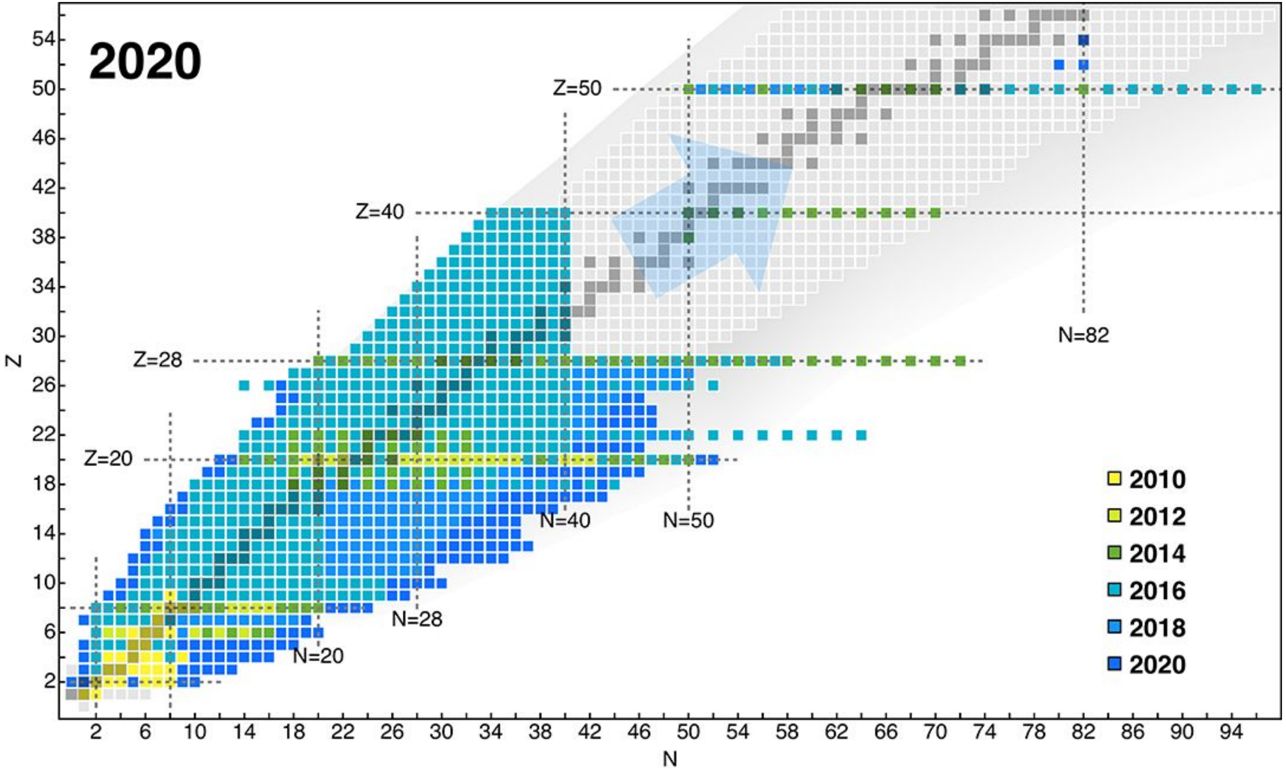
Ψ are spin-isospin vectors in 3A dimensions with $2^A \times \frac{A!}{Z!(A-Z)!}$ components

^{4}He : 96
^{6}Li : 1280
^{8}Li : 14336
^{12}C : 540572

Develop Computational Methods to solve (numerically) exactly or within approximations that are under control the many-body nuclear problem

$$H\Psi = E\Psi$$

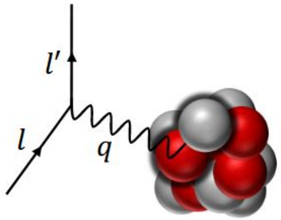
Current Status



H. Hergert
Front. Phys.
07 October 2020

Challenges in Lepton-Nucleus Scattering

- Extend the *ab initio* approach to include scattering processes
- Extend the *ab initio* to address heavier nuclei of experimental interest
- Incorporate relativistic effects
- Study inelastic processes
- …



Quantum Monte Carlo Methods

Minimize the expectation value of the nuclear Hamiltonian: H = T +v_{ij} + V_{ijk}

$$E_V = \frac{\langle \Psi_V | H | \Psi_V \rangle}{\langle \Psi_V | \Psi_V \rangle} \geq E_0$$

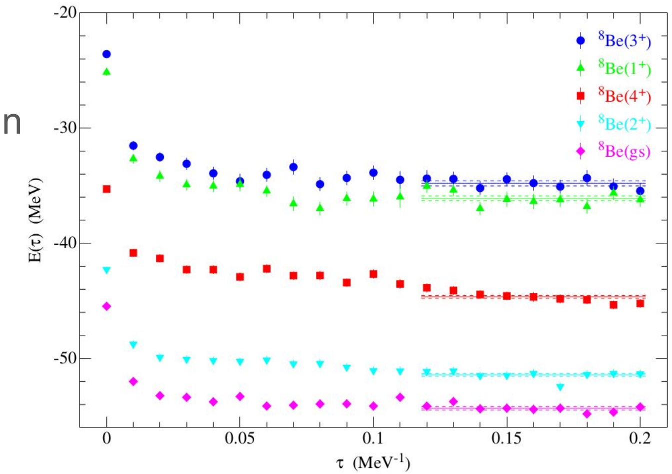
using the trial wave function:

$$|\Psi_V\rangle = \left[\mathcal{S} \prod_{i<j} (1 + U_{ij} + \sum_{k \neq i,j} U_{ijk}) \right] \left[\prod_{i<j} f_c(r_{ij}) \right] |\Phi_A(JMTT_3)\rangle$$

Further improve the trial wave function by eliminating spurious contaminations via a Green's Function Monte Carlo propagation in imaginary time

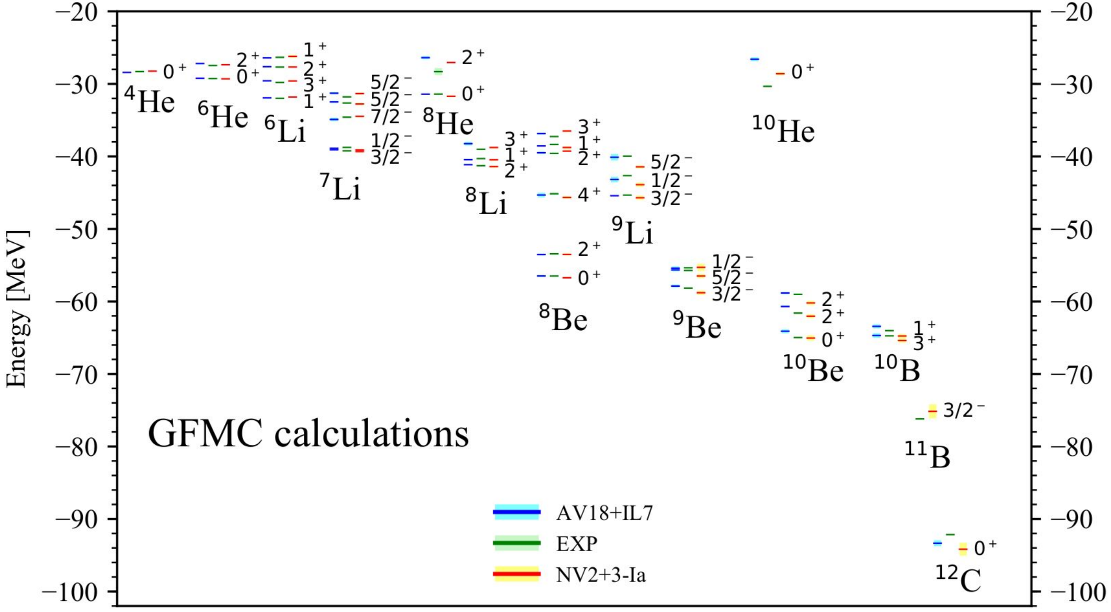
$$\Psi(\tau) = \exp[-(H - E_0)\tau]\Psi_V = \sum_n \exp[-(E_n - E_0)\tau] a_n \psi_n$$

$$\Psi(\tau \to \infty) = a_0 \psi_0$$

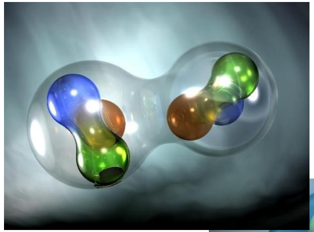


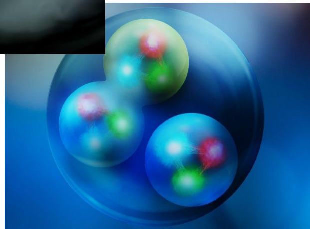
Carlson, Wiringa, Pieper *et al.*

Energies



Piarulli *et al.* PRL120(2018)052503





The Yukawa potential

Yukawa potential (1930s): Yukawa potential is due to the coupling of the interacting nucleons to a massive field.

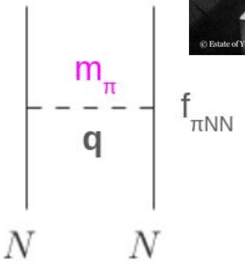
$$v_Y \sim -\frac{e^{-\mu r_{ij}}}{r_{ij}} \qquad \text{range} \propto \frac{1}{\mu}$$

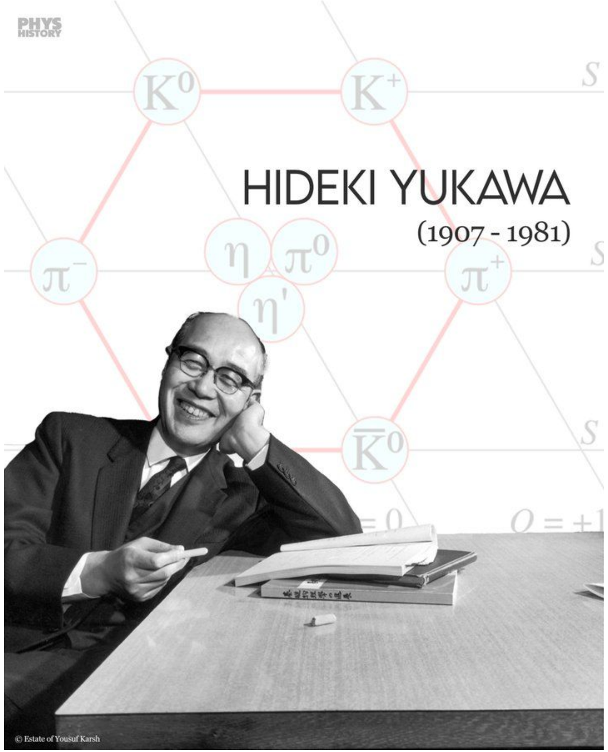
The pion is observed in 1947: With $\mu = m_\pi \sim 140$ MeV implying a range ~ 1.4 fm.

$$v_{12}^{\pi}(\mathbf{r}) = \frac{f_{\pi NN}^2}{4\pi}\frac{m_\pi}{3}\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 \left\{ T_\pi(r) S_{12} + \left[Y_\pi(r) - \frac{4\pi}{m_\pi^3}\delta(\mathbf{r}) \right] \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \right\}$$

Pion-nucleon coupling constant

Tensor operator



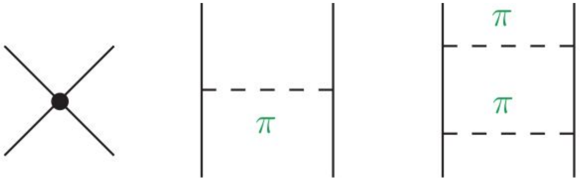


Many-body Nuclear Interactions

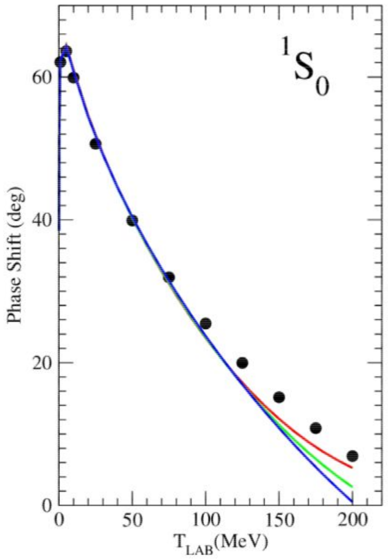
Many-body Nuclear Hamiltonian

$$H = T + V = \sum_{i=1}^{A} t_i + \sum_{i<j} v_{ij} + \sum_{i<j<k} V_{ijk} + \ldots$$

v_{ij} and V_{ijk} are two- and three-nucleon operators based on experimental data fitting; fitted parameters subsume underlying QCD dynamics



Contact term: short-range
Two-pion range: intermediate-range $r \propto (2\,m_\pi)^{-1}$
One-pion range: long-range $r \propto m_\pi^{-1}$



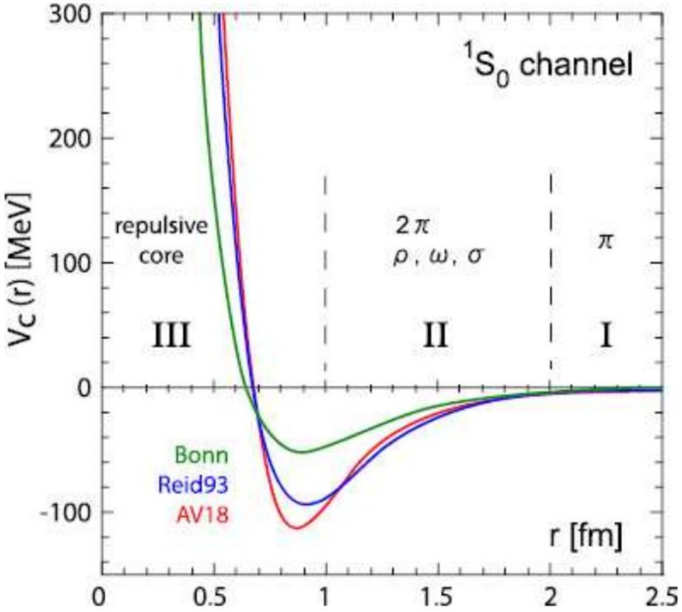
SP et al. PRC80(2009)034004

Hideki Yukawa

AV18+UIX; AV18+IL7
Wiringa, Schiavilla, Pieper *et al*.

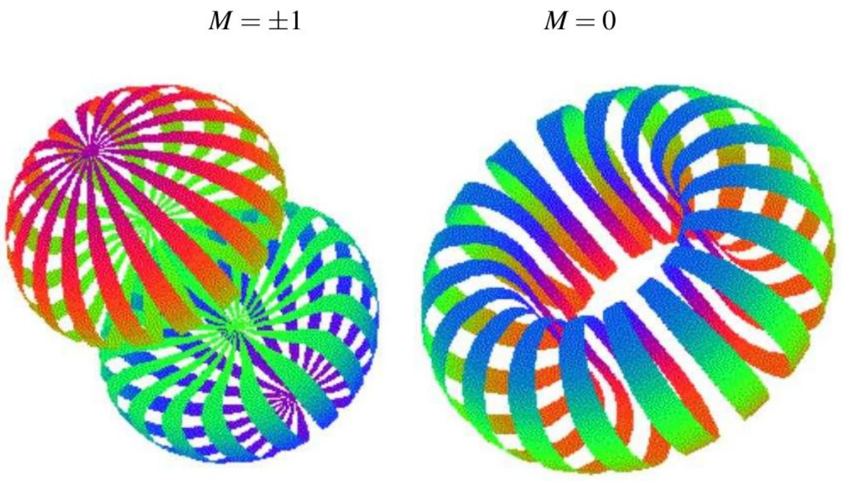
chiral πNΔ
N3LO+N2LO Piarulli *et al*. **Norfolk Models**

Nucleon-Nucleon Potential



Aoki *et al.* Comput.Sci.Disc.1(2008)015009

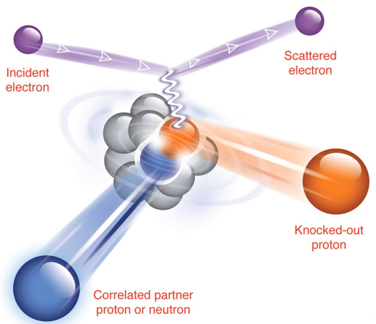
The Deuteron

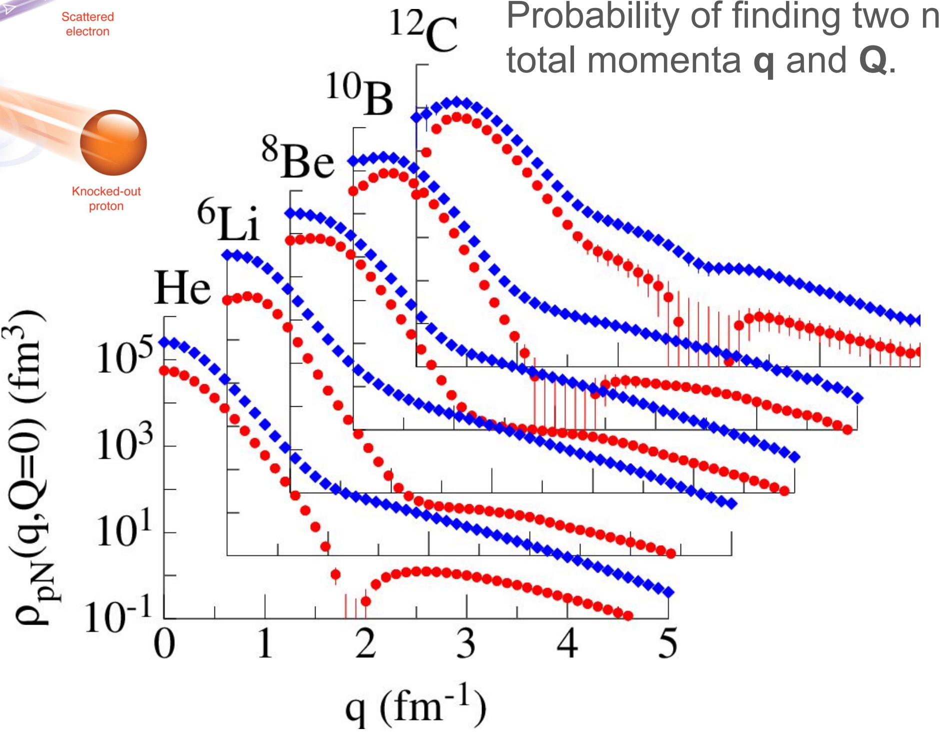


Constant density surfaces for a polarized deuteron in the $M = \pm 1$ (left) and $M = 0$ (right) states

Carlson and Schiavilla Rev.Mod.Phys.70(1998)743

Two-nucleon correlations & momentum distributions





pp-pairs; np-pairs

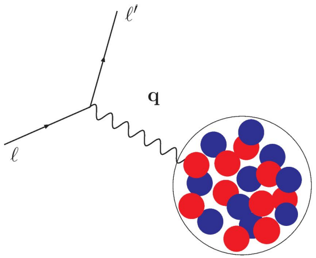
Probability of finding two nucleons with relative and total momenta **q** and **Q**.

Tensor correlations lead to large differences in the np versus pp distributions.

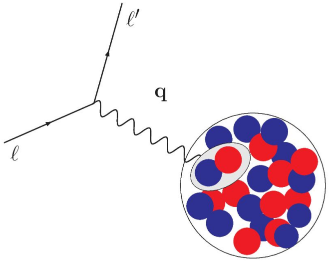
These differences are observed in A(e, e′np) and A(e, e′pp) reactions.

Schiavilla Carlson Wiringa Pieper
PRL98(2007) & PRC89(2014)

Many-body Nuclear Electroweak Currents



one-body



two-body

- Two-body currents are a manifestation of two-nucleon correlations
- Electromagnetic two-body currents are required to satisfy current conservation

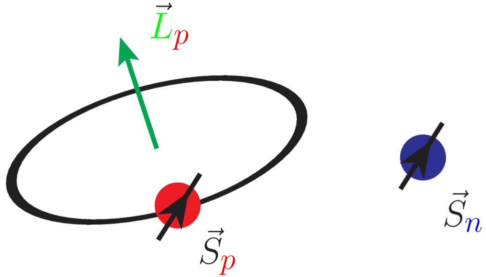
$$\mathbf{q}\cdot\mathbf{j} = [\,H,\,\rho\,] = [t_i + v_{ij} + V_{ijk},\,\rho\,]$$

Nuclear Charge Operator

$$\rho = \sum_{i=1}^{A} \rho_i + \sum_{i<j} \rho_{ij} + \ldots$$

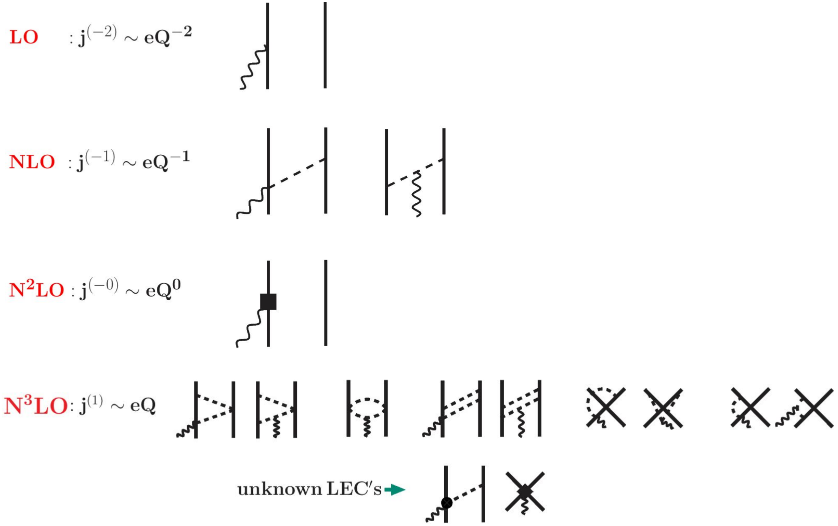
Nuclear (Vector) Current Operator

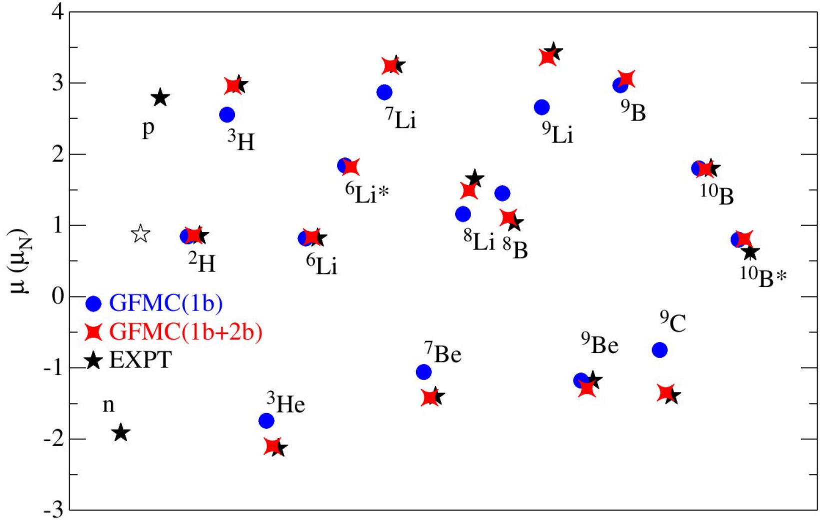
$$\mathbf{j} = \sum_{i=1}^{A} \mathbf{j}_i + \sum_{i<j} \mathbf{j}_{ij} + \ldots$$



Magnetic Moment: Single Particle Picture

Many-body Currents



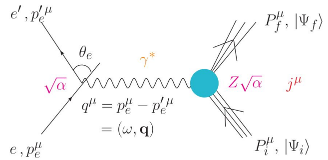


SP *et al*. PRC87(2013)035503

Electromagnetic Current Operator

SP *et al.* PRC78(2008)064002, PRC80(2009)034004, PRC84(2011)024001, PRC87(2013)014006
Park *et al.* NPA596(1996)515, Phillips (2005), Kölling *et al.* PRC80(2009)045502, PRC84(2011)054008

Elastic scattering and the shape of nuclei



Cross section

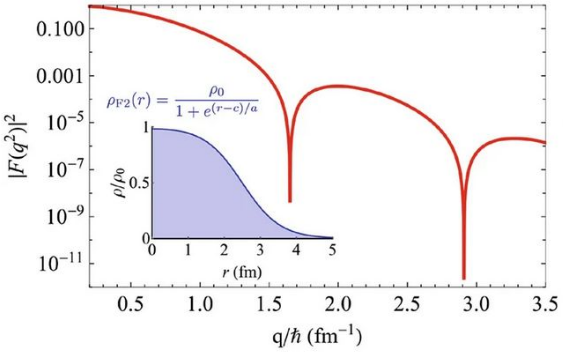
$$\frac{d\sigma}{d\Omega} = 4\pi\sigma_M f_{\text{rec}}^{-1}\left[\frac{Q^4}{q^4}F_L^2(q) + \left(\frac{Q^2}{2q^2} + \tan^2\theta_e/2\right)F_T^2(q)\right]$$

Magnetic Form Factor

$$F_T^2(q) = F_M^2(q) = \frac{1}{2J_i+1}\sum_{L=1}^{\infty}|\langle J_f||M_L(q)||J_i\rangle|^2 \quad \propto \quad \langle JJ|\mathrm{j}_y(q\hat{\boldsymbol{x}})|JJ\rangle$$

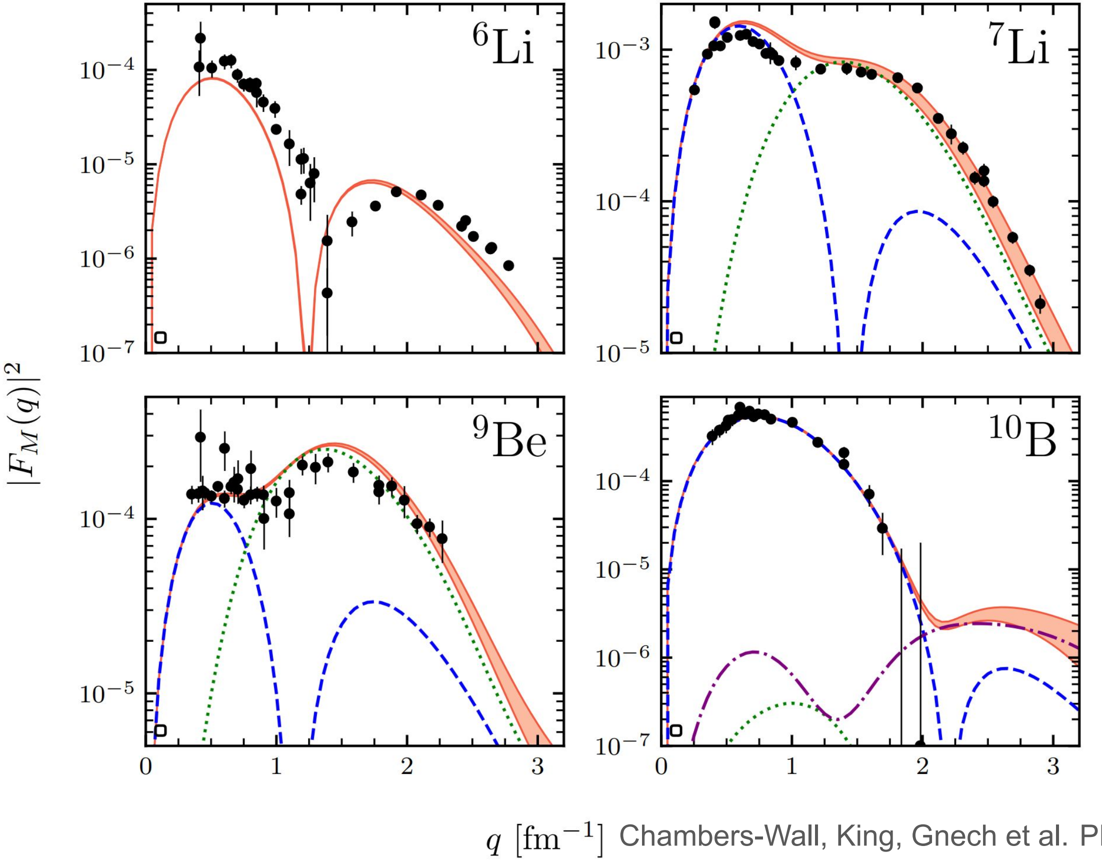
Charge Form Factors

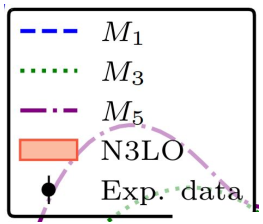
$$F_L^2(q) = \frac{1}{2J_i+1}\sum_{L=0}^{\infty}|\langle J_f||C_L(q)||J_i\rangle|^2 \quad \propto \quad \langle J_f M|\rho^\dagger(q)|J_i M\rangle$$



Nörtershäuser 2020

Magnetic form factors: comparison with the data





Nucleus	Reference	Data type	ratio/method
^{3}H	Sick 2001 [89]	N	1
^{3}He	Sick 2001 [89]	N	1
^{6}Li	Peterson 1962 [90]	N	Eq. (C2)
	Goldemberg 1963 [91]	N	Eq. (C2)
	Rand 1966 [92]	N	Eq. (C1)
	Lapikas 1978 [93]	D	$1/4\pi$
	Bergstrom 1982 [94]	N	$Z^2/4\pi$
^{7}Li	Peterson 1962 [90]	N	Eq. (C2)
	Goldemberg 1963 [91]	N	Eq. (C2)
	Van Niftrik 1971 [95]	D	Eq. (C1)
	Lichtenstadt 1983 [96]	N	$Z^2/4\pi$
^{9}Be	Goldemberg 1963 [91]	N	Eq. (C2)
	Vanpraet 1965 [98]	N	Eq. (C1)
	Rand 1966 [92]	N	Eq. (C1)
	Lapikas 1975 [97]	N	Eq. (C2)
^{10}B	Goldemberg 1963 [91]	N	Eq. (C2)
	Goldemberg 1965 [100]	N	Eq. (C2)
	Vanpraet 1965 [98]	N	Eq. (C1)
	Rand 1966 [92]	N	Eq. (C1)
	Lapikas 1978 [93]	D	$1/4\pi$

Chambers-Wall, King, Gnech et al. PRL 2024 2407.03487

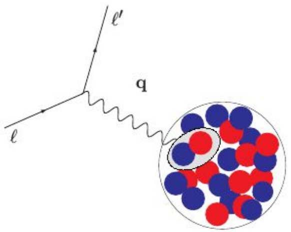
Lepton-Nucleus scattering: Inclusive Processes

Electromagnetic Nuclear Response Functions

$$R_\alpha(q,\omega) = \sum_f \delta\left(\omega + E_0 - E_f\right) |\langle f|O_\alpha(\mathbf{q})|0\rangle|^2$$

Longitudinal response induced by the charge operator $O_L = \rho$
Transverse response induced by the current operator $O_T = \boldsymbol{j}$
5 Responses in neutrino-nucleus scattering

$$\frac{d^2\sigma}{d\omega d\Omega} = \sigma_M \left[v_L R_L(\mathbf{q},\omega) + v_T R_T(\mathbf{q},\omega)\right]$$

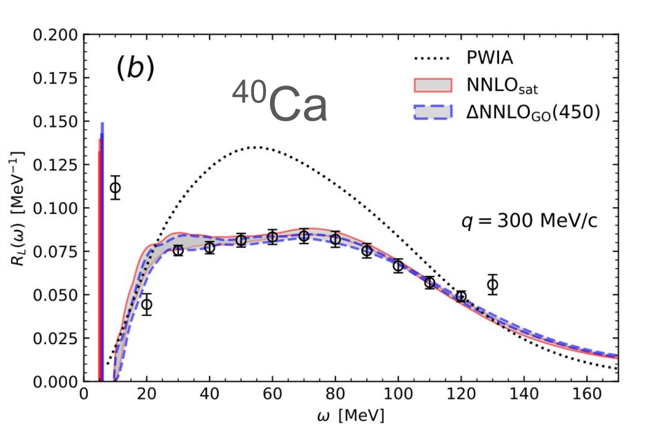


For a recent review on QMC, SF methods see
Rocco *Front. In Phys.*8 (2020)116

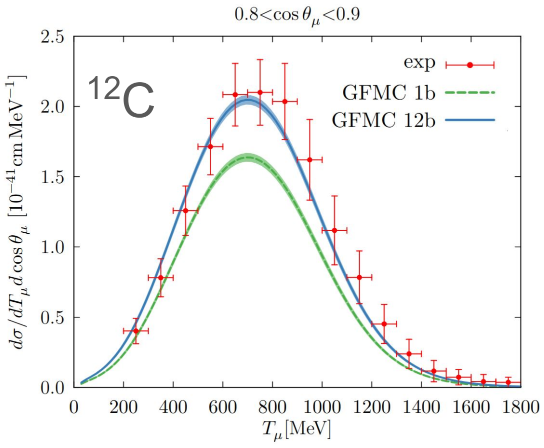
Inclusive Cross Sections with Integral Transforms

Exploit integral properties of the response functions and closure to avoid explicit calculation of the final states (Lorentz Integral Transform **LIT**, **Euclidean**, …). Limited to inclusive processes.

$$S(q,\tau) = \int_0^\infty d\omega\, K(\tau,\omega) R_\alpha(q,\omega)$$



Sobczyk et al, PRL127 (2021)



Lovato et al. PRX10 (2020)

Lepton-Nucleus Scattering

QMC's effort has been extensively addressed to study inclusive QE electroweak processes

Nuclear Response Function

$$R_\alpha(q,\omega) = \overline{\sum_{M_i}} \sum_f \langle \Psi_i | O_\alpha^\dagger(\mathbf{q}) | \Psi_f \rangle \langle \Psi_f | O_\alpha(\mathbf{q}) | \Psi_i \rangle \, \delta(E_f - E_i - \omega)$$

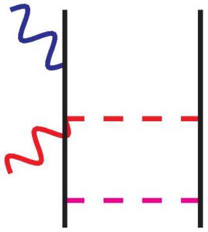
Longitudinal Response $O^{(L)}(\mathbf{q}) = \rho(\mathbf{q})$

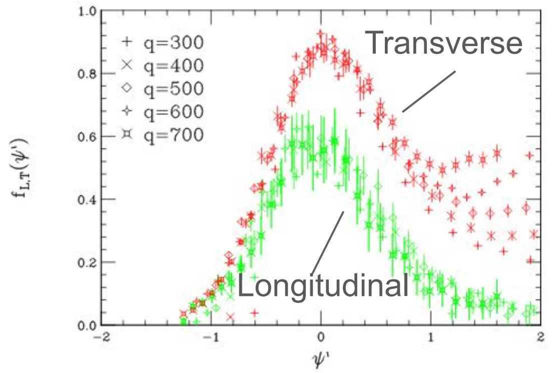
Transverse Response $O^{(T)}(\mathbf{q}) = \mathbf{j}(\mathbf{q})$

} Electron scattering

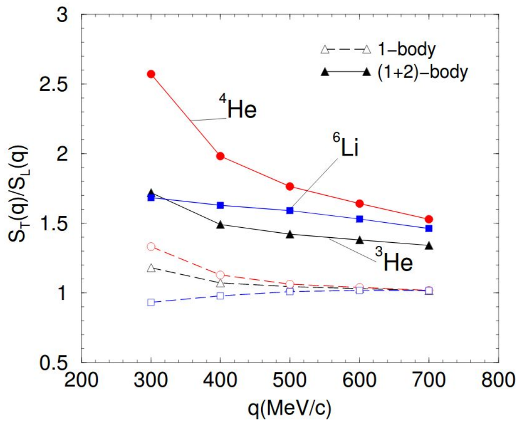
Two-nucleon correlations and currents required to explain electron scattering data in the QE region, including the *Interference* term

one + two-body interference $\langle \mathbf{j}_i \, \mathbf{j}_{ij} \, v_{ij} \rangle > 0$





^{4}He Electromagnetic Data



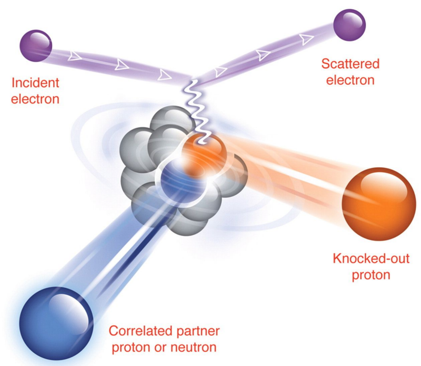
Transverse/Longitudinal

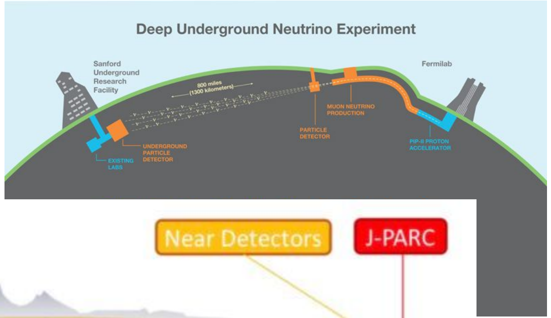
Carlson *et al.* PRC65(2002)024002

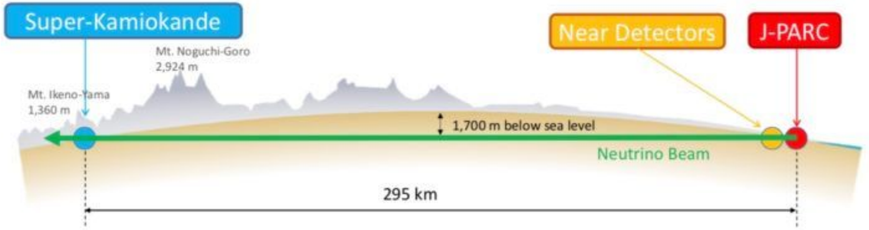
Scattering with the Short-Time-Approximation

Short-Time-Approximation Goals:

- Describe electroweak scattering from A > 12 without losing two-body physics
- Account for *exclusive processes*
- Incorporate *relativistic effects*



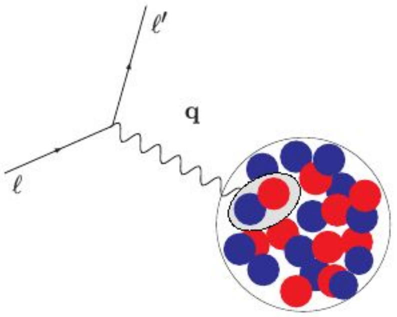




Subedi et al. Science320(2008)1475

e4u collaboration

Short-Time-Approximation



Short-Time-Approximation:

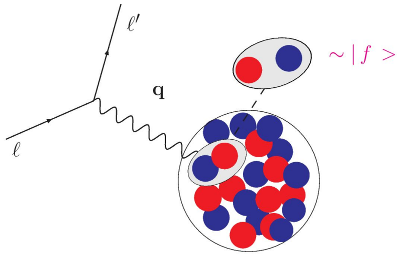
- Based on Factorization
- Retains two-body physics at the vertex
- Correctly accounts for interference

$$R(q,\omega) = \int_{-\infty}^{\infty} \frac{dt}{2\pi}\, \mathrm{e}^{i(\omega+E_0)t}\, \langle 0|O^\dagger\, \mathrm{e}^{-iHt}\, O|0\rangle$$

$$O_i^\dagger\, \mathrm{e}^{-iHt}\, O_i + O_i^\dagger\, \mathrm{e}^{-iHt}\, O_j + O_i^\dagger\, \mathrm{e}^{-iHt}\, O_{ij} + O_{ij}^\dagger\, \mathrm{e}^{-iHt}\, O_{ij}$$

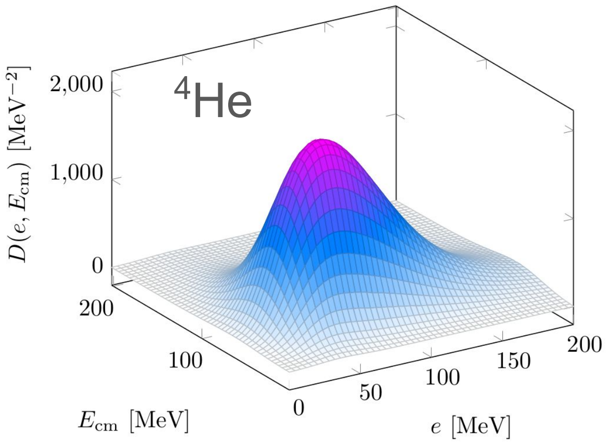
$$H \sim \sum_i t_i + \sum_{i<j} v_{ij}$$

Short-Time-Approximation



Short-Time-Approximation:

- Based on Factorization
- Retains two-body physics
- Accounts for interference term



SP et al. PRC101(2020)044612

Response Functions ∝ Cross Sections

$$R_\alpha(q,\omega) = \sum_f \delta\left(\omega + E_0 - E_f\right) |\langle f|O_\alpha(\mathbf{q})|0\rangle|^2$$

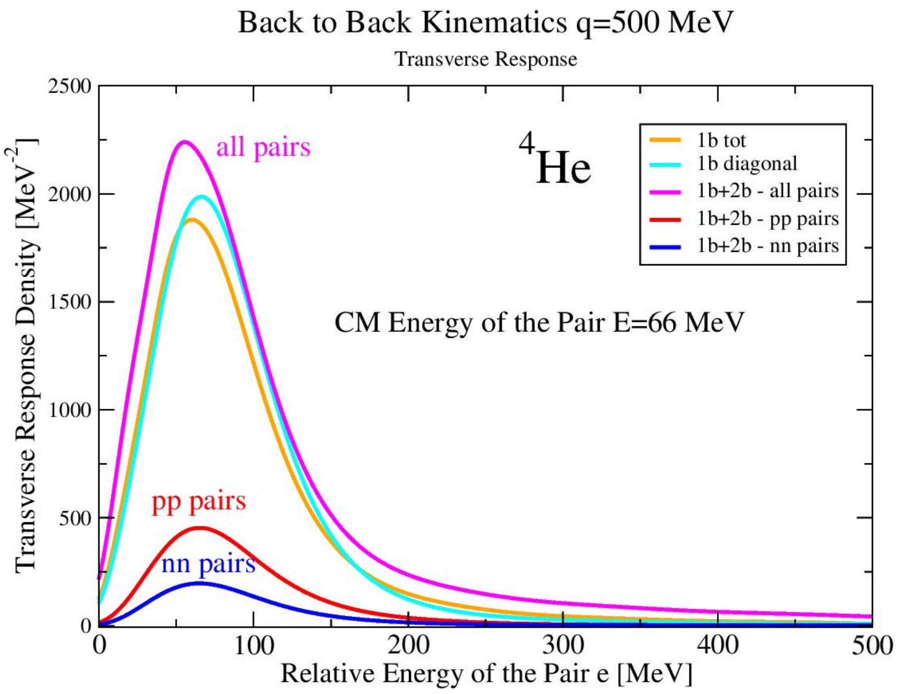
Response ***Densities***

$$R(q,\omega) \sim \int \delta\left(\omega + E_0 - E_f\right)\, dP'\, dp' \mathcal{D}(p', P'; q)$$

P' and *p'* are the CM and relative momenta of the struck nucleon pair

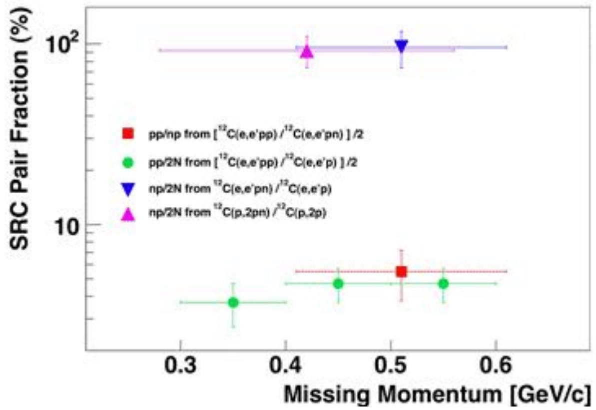
Response Densities provide “more” exclusive information about the struck nucleon-pair

e-^{4}He scattering in the back-to-back kinematic



SP *et al.* PRC101(2020)044612

- pp pairs
- nn pairs
- all pairs 1 body
- all pairs tot



Subedi et al. Science 320(2008)

STA: regime of validity

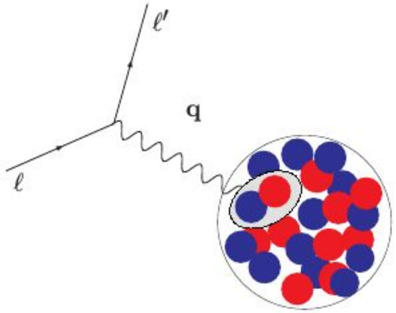
The typical (conservative estimate) energy (time) scale in a nucleus with A correlated nucleons in pairs is

$\varepsilon_{pair} \sim 20$ MeV   ($t \sim 1/\varepsilon_{pair}$)

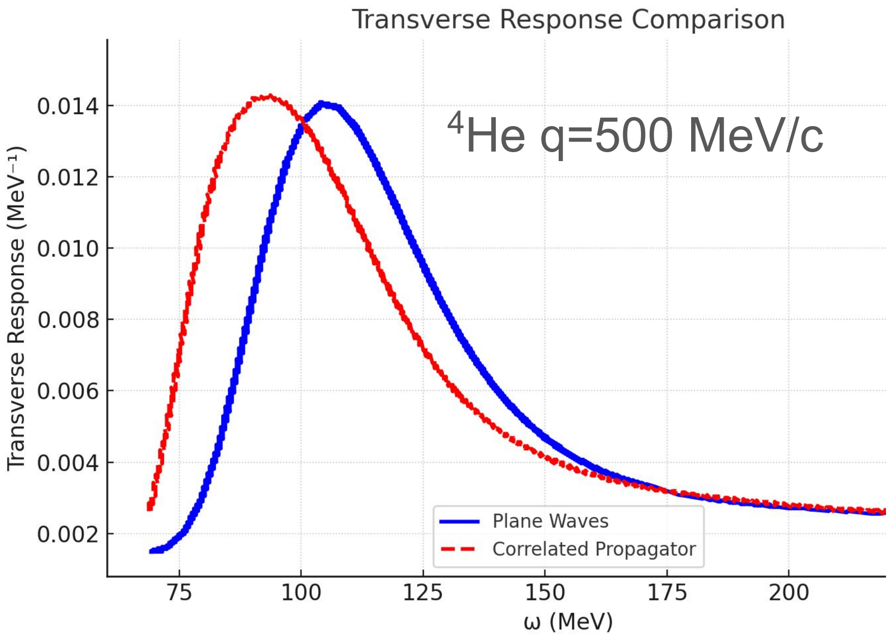
This sets a natural expansion parameter in the QE region characterized by ω_{QE}

$\varepsilon_{pair} / \omega_{QE}$

The STA neglects terms of order $\mathcal{O}((\varepsilon_{pair} / \omega_{QE})^2)$

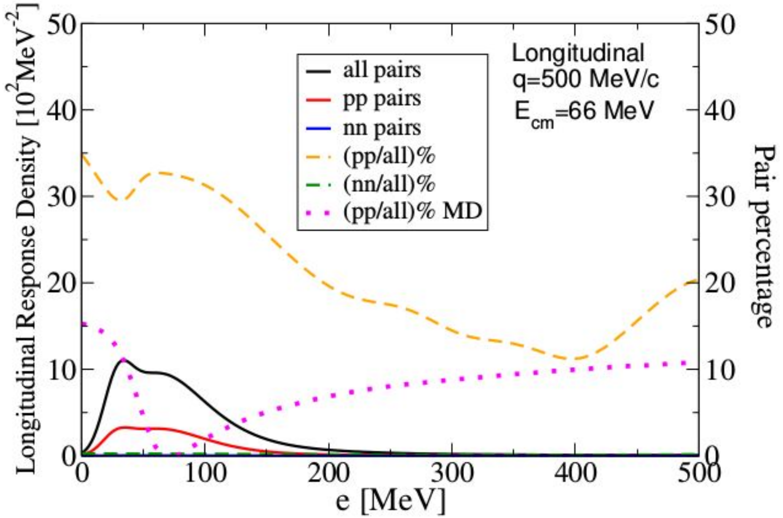


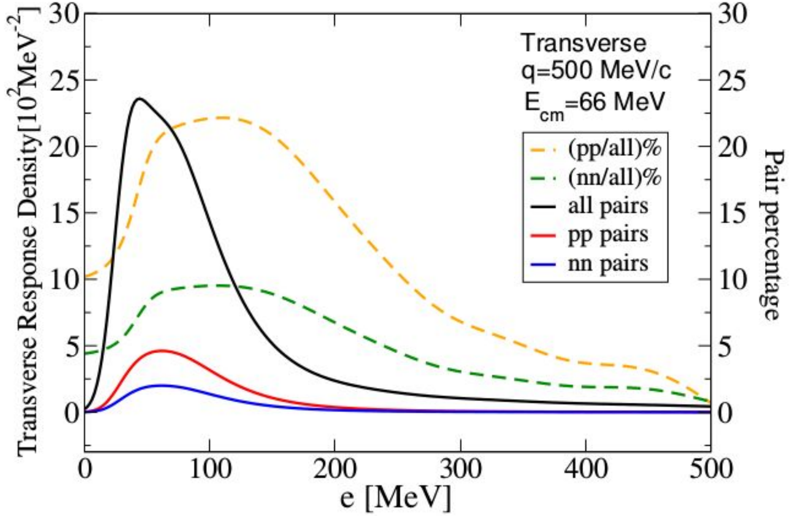
Correlated pairs vs uncorrelated pairs



Scattering from uncorrelated vs correlated nucleon pairs

Back to back scattering and particle identity





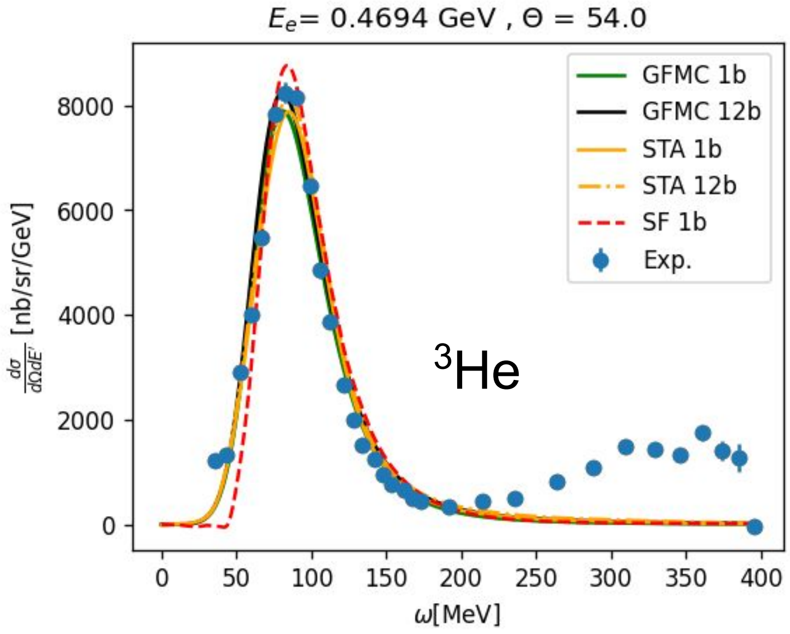
tot

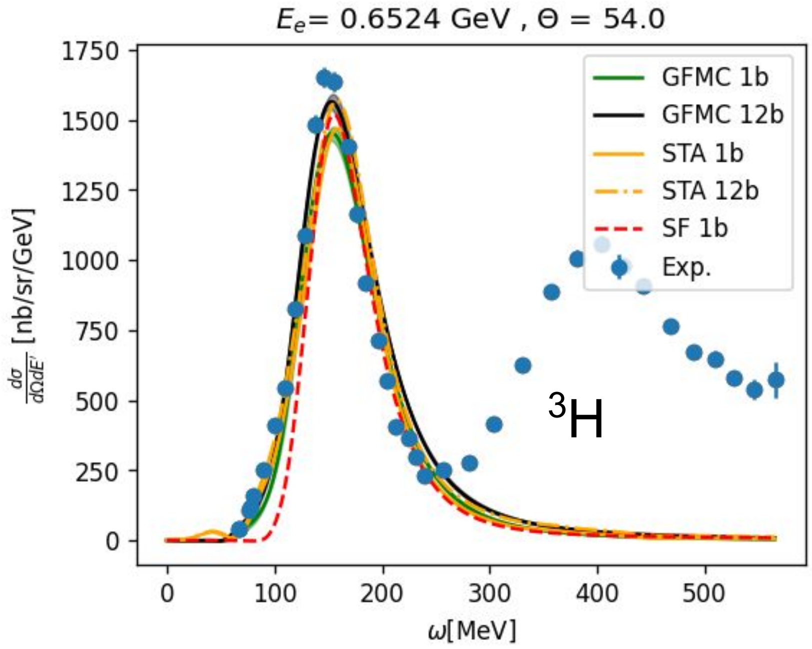
pp nn

pp/all % pp/all % from momentum distributions

nn/all %

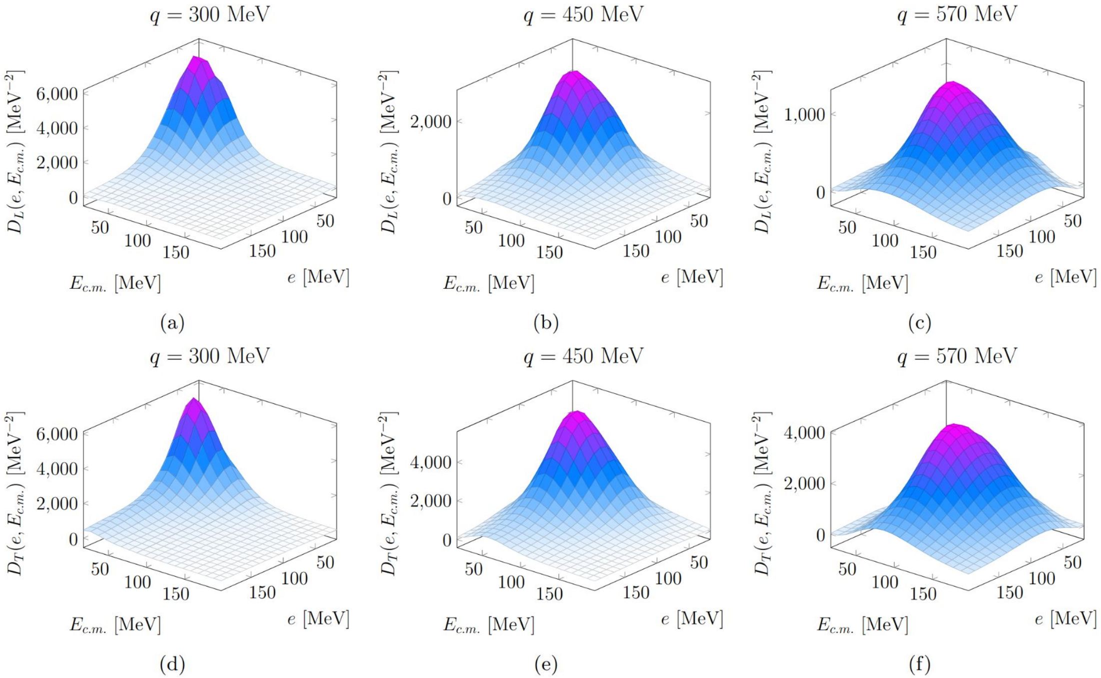
GFMC SF STA: Benchmark & error estimate in A=3





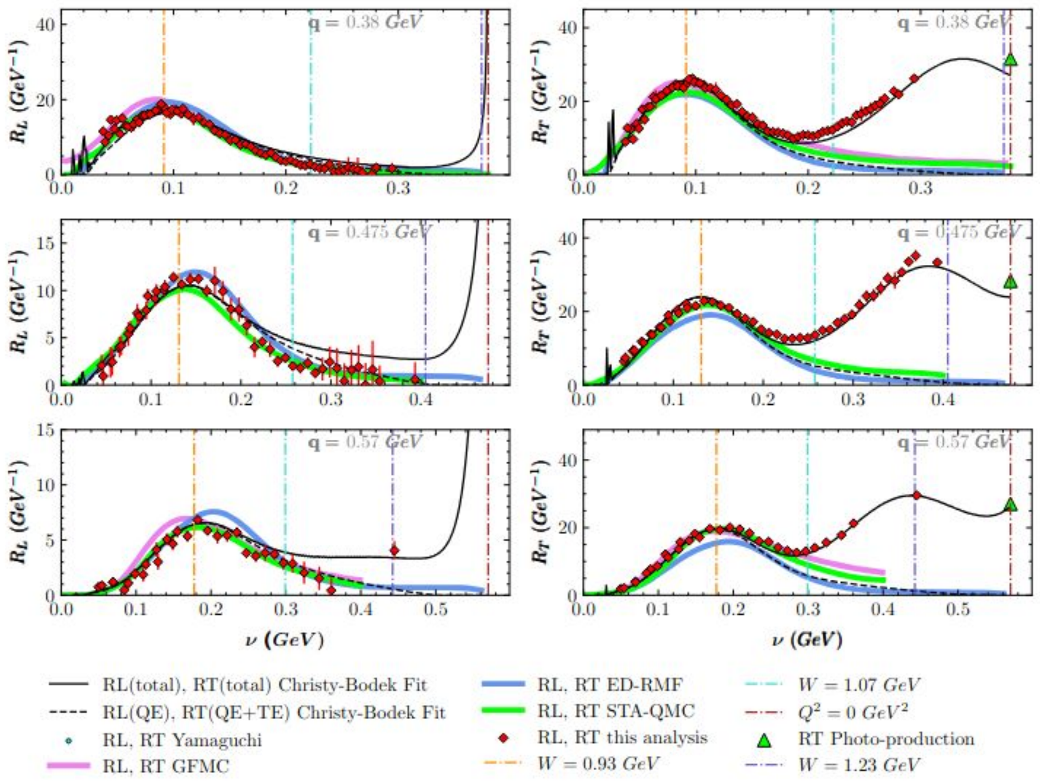
Andreoli *et al.* PRC 105 (2022)

^{12}C Response Densities



Andreoli, SP *et al.* PRC110(2024)

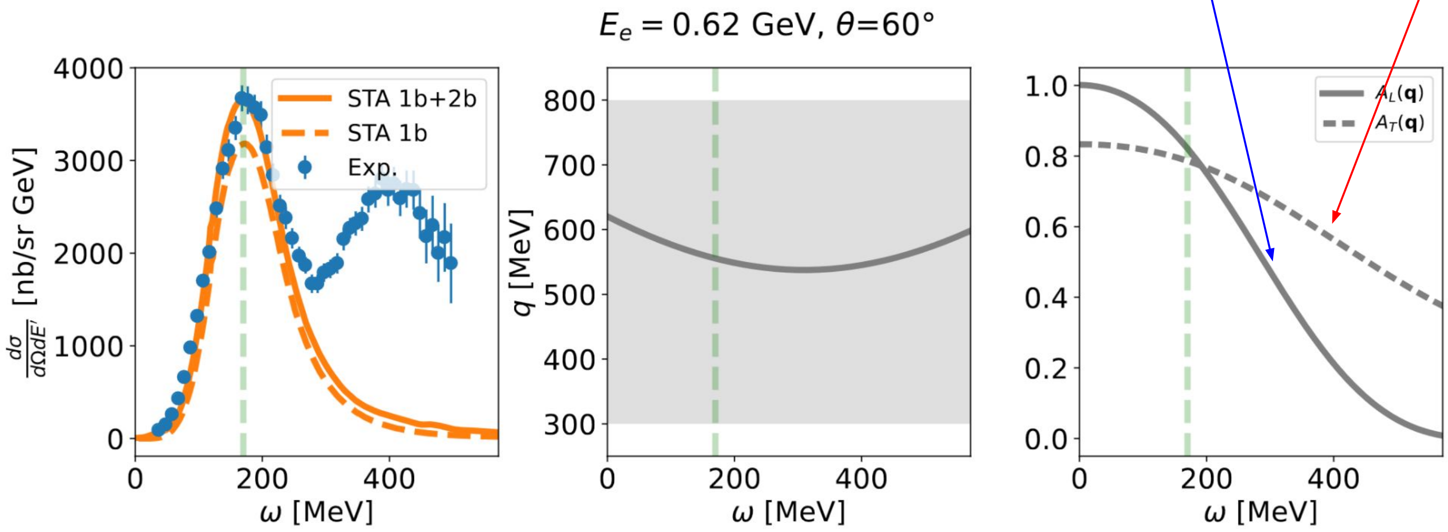
^{12}C comparison with the data



Christy, Bodek et al (2025)

^{12}C cross sections

$$\frac{d^2\sigma}{d\omega d\Omega} = \sigma_M \left[v_L R_L(\mathbf{q},\omega) + v_T R_T(\mathbf{q},\omega)\right]$$

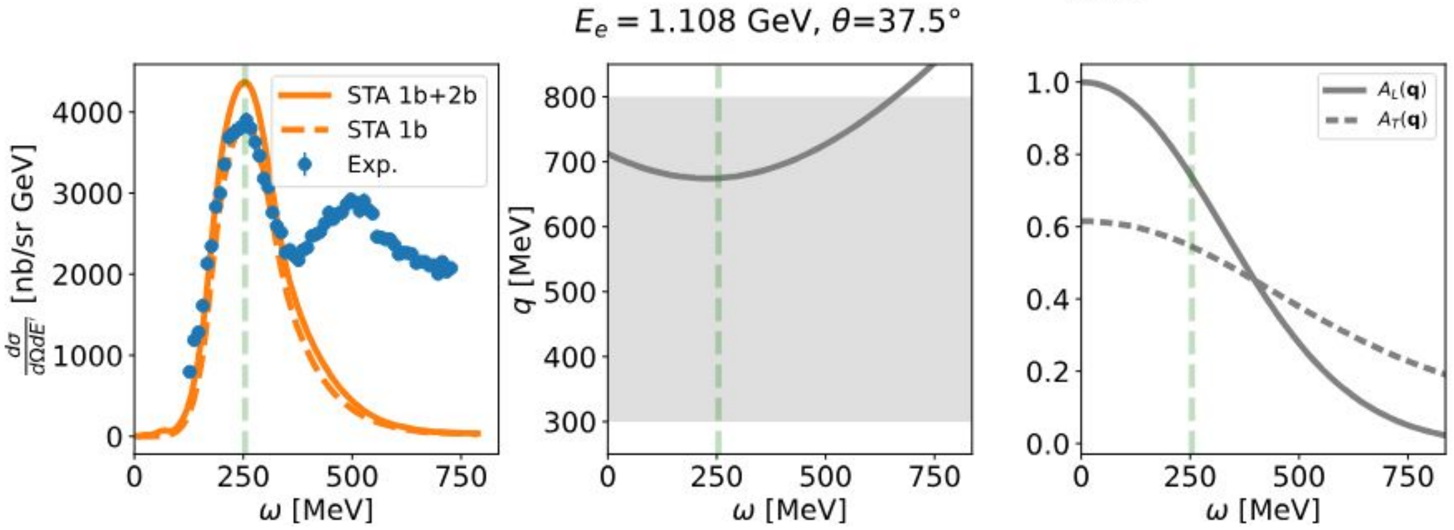


Andreoli *et al.* *Phys.Rev.C* 110 (2024) 6, 064004 arXiv:2407.06986

Data From https://discovery.phys.virginia.edu/research/groups/qes-archive/index.html

Relativistic effects in e-^{12}C scattering

$$\frac{d^2\sigma}{d\omega d\Omega} = \sigma_M \left[v_L R_L(\mathbf{q},\omega) + v_T R_T(\mathbf{q},\omega) \right]$$



Andreoli *et al.* *Phys.Rev.C* 110 (2024) 6, 064004 arXiv:2407.06986

Relativistic corrections: Operator

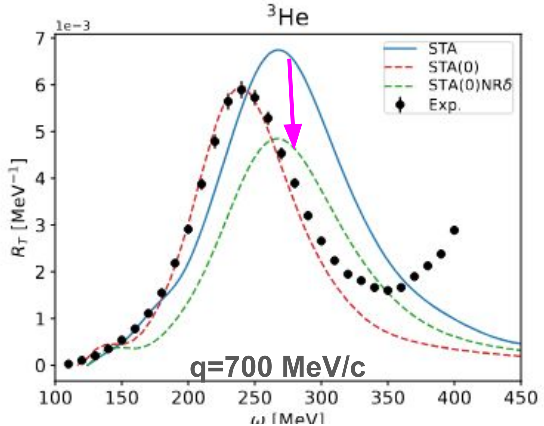
$$j^\mu = e\bar{u}(\boldsymbol{p}'s')\left(e_N\gamma^\mu + \frac{i\kappa_N}{2m_N}\sigma^{\mu\nu}q_\nu\right)u(\boldsymbol{p}s)$$

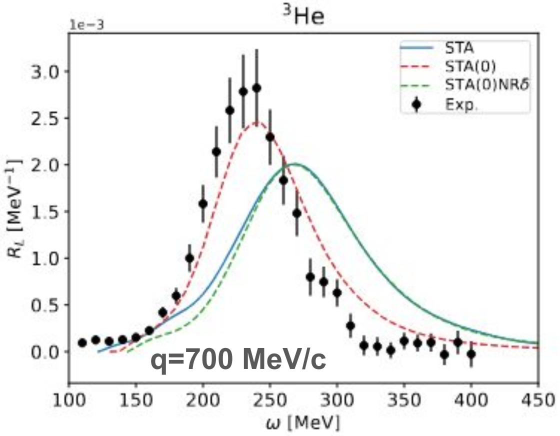
$$\boldsymbol{p}' = \boldsymbol{p} + \boldsymbol{q}$$

We updated the single nucleon current operators resulting from the non-relativistic expansion of the covariant single-nucleon current

$$\frac{p}{m} \simeq \frac{p'}{m} \simeq \frac{q}{m} \ll 1$$ →valid at low q

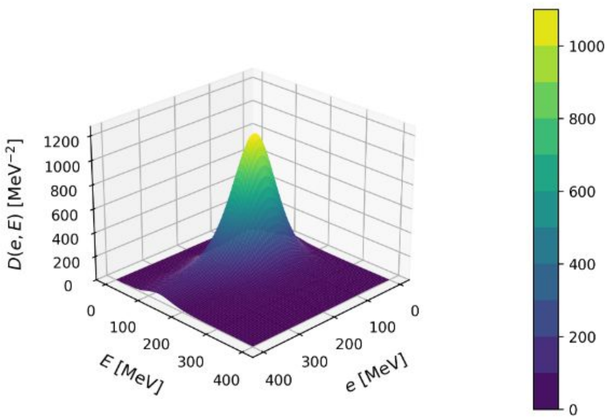
$$\frac{p}{m} \ll 1$$ →valid at any q;
no assumption on the nucleon' final momentum





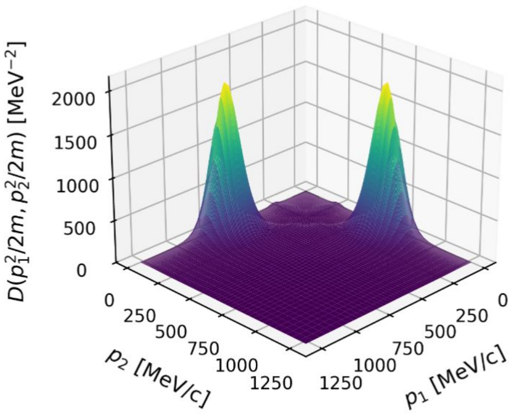
Lorenzo Andreoli, Ronen Weiss, et al. PRC113 2025

Relativistic corrections: Final state kinematics

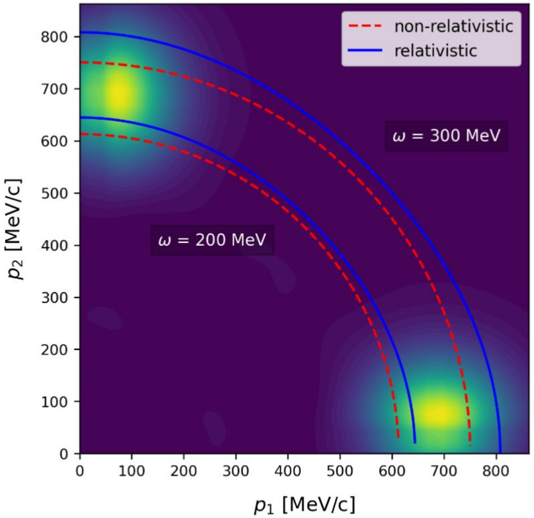


Response density vs relative and c.m. energy of the struck pair

^{4}He Transverse response density at q = 700 MeV/c



Response density vs momenta of individual nucleons



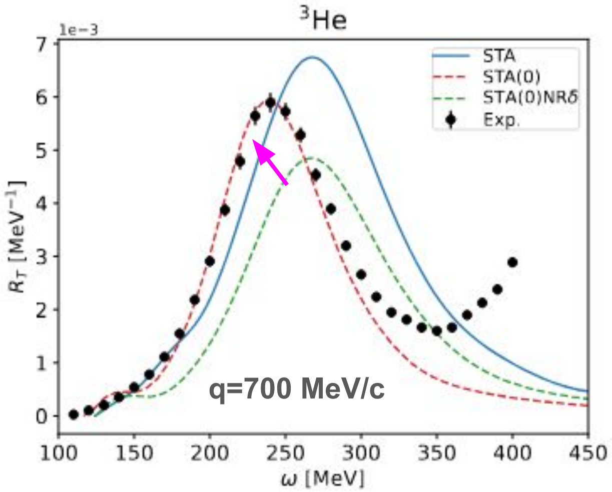
$$R_\alpha(q,\omega) = \sum_f \delta\left(\omega + E_0 - E_f\right) |\langle f|O_\alpha(\mathbf{q})|0\rangle|^2$$

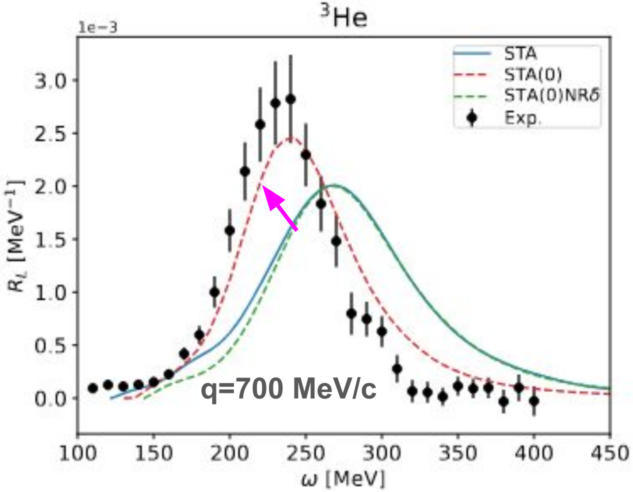
Relativistic kinematics

Non-relativistic kinematics

Lorenzo Andreoli, Ronen Weiss, et al. PRC113 2025

Relativistic corrections: Final state kinematics

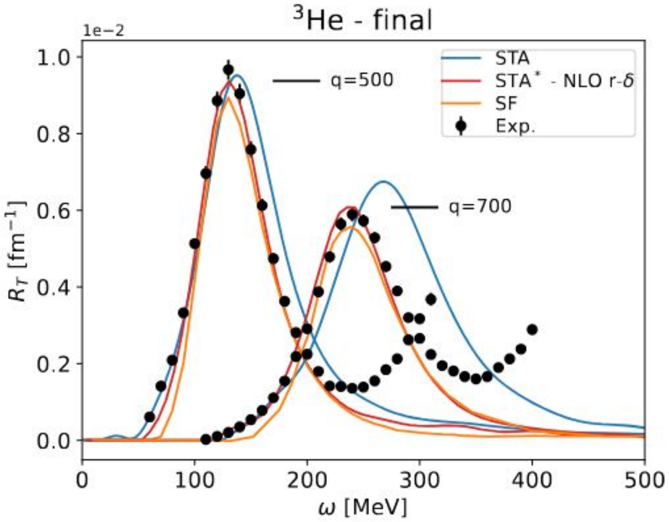




Relativistic treatment of two-nucleon final states kinematic improve description of data at high-energies

Lorenzo Andreoli, Ronen Weiss, et al. PRC113 2025

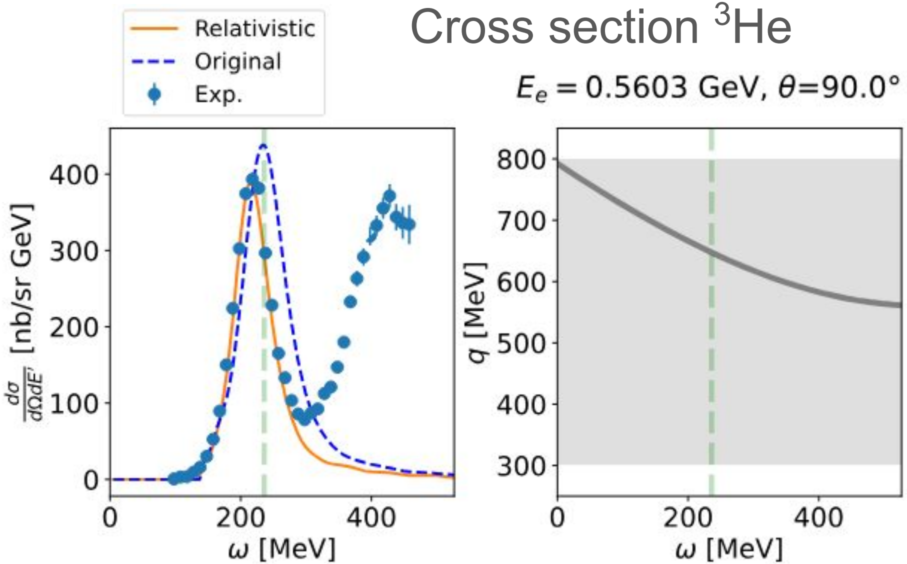
Incorporating relativistic corrections at the vertex



Covariant single nucleon currents expanded in p/m (p = initial nucleon momentum)

Relativistic two-nucleon final states kinematics included

Cross section ^{3}He



Lorenzo Andreoli, Ronen Weiss, et al.

PRC113 2025

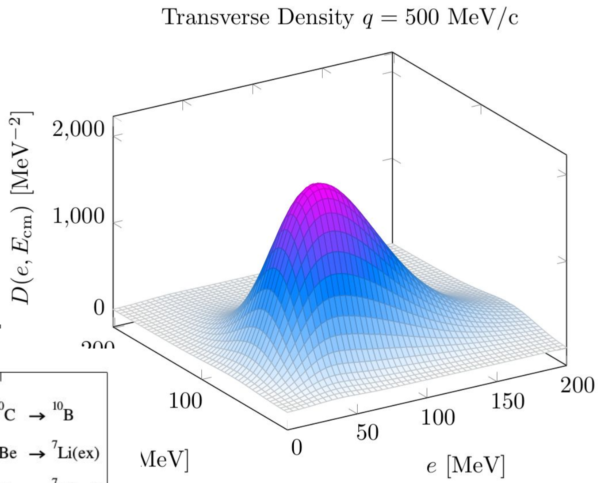
Summary & Outlook

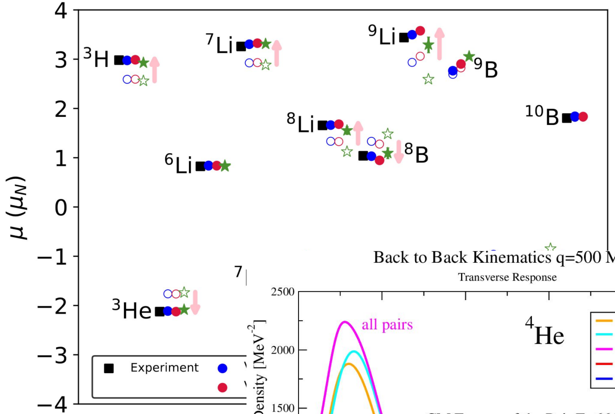
- STA+VMC applied to study scattering from A=3, 4, 12 systems
  - Good agreement with data and previous theoretical calculations
  - Relativistic corrections (RC) required to explain data

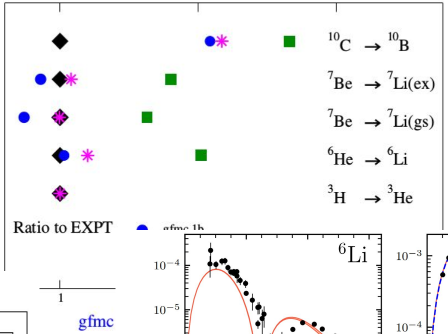
- Studies of ^{12}C with RC and other light nuclei
- Incorporate angular dependence in the STA response densities
- Two-nucleon momentum distribution with angular dependence in progress by Ronen Weiss *et al* (this morning's talk)
- Exporting the STA in AFDMC to allow studies in A>12 systems
- Incorporate inelasticity (e.g., pion production)

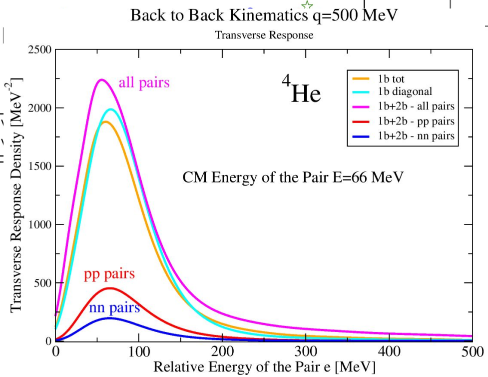
Summary

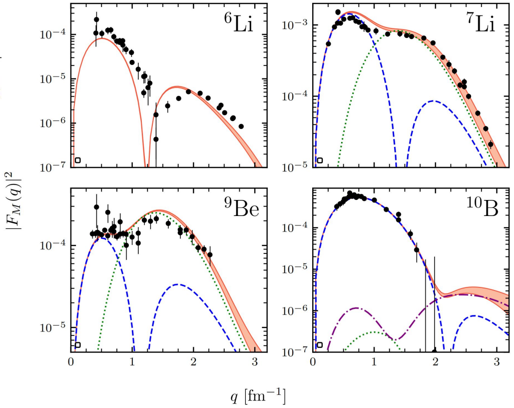
Ab initio calculations of light nuclei yield a picture of nuclear structure and dynamics where many-body effects play an essential role to explain available data.











Close **collaborations** between **NP**, **LQCD**, **Pheno**, **Hep**, **Comp**, **Expt**, … are required to progress e.g., NP is represented in the Snowmass process

It's a very exciting time!

Collaborators

WashU: **Chambers-Wall Macedo-Lima Dawid Piarulli Weiss**
LANL: Carlson Gandolfi Hayes **King** Mereghetti Flores
JLab+ODU: **Andreoli Gnech** Schiavilla
ANL: McCoy Lovato Wiringa
UW/INT: Cirigliano Dekens
Pisa U/INFN: Kievsky Marcucci Viviani
Salento U: Girlanda
Huzhou U: Dong Wang
Fermilab: Gardiner Betancourt Rocco
MIT: Barrow

Washington University in St. Louis
Arts & Sciences

U.S. DEPARTMENT OF ENERGY
INCITE
LEADERSHIP COMPUTING

NUCLEI
Nuclear Computational Low-Energy Initiative
A SciDAC-5 Project

NTNP

ASCR Leadership Computing Challenge
ALCC
2024-2025 Awards

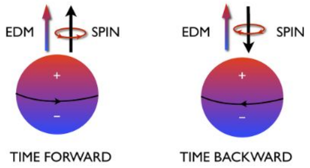
U.S. DEPARTMENT OF ENERGY | Office of Science

McDONNELL CENTER
FOR THE SPACE SCIENCES

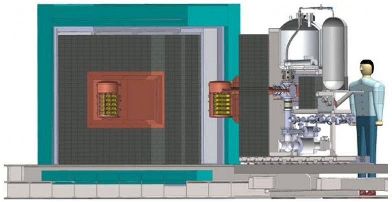
Fermilab

NERSC

Ground States' Electroweak Moments, Form Factors, Radii



Neutrinoless Double Beta Decay, Muon-Capture



Accelerator Neutrino Experiments, Lepton-Nucleus XSecs

Jefferson Lab

(ω,q)~0 MeV

ω~few MeVs q~0 MeV

ω~few MeVs $q\sim10^2$ MeV

ω~tens of MeVs

$\omega\sim10^2$ MeV

FRIB

Argonne NATIONAL LABORATORY

RIKEN NiSHiNA CENTER

GANiL

Electromagnetic Decay, Beta Decay, Double Beta Decay & inverse processes

JINA-CEE

Nuclear Rates for Astrophysics

DUNE

T2K

Graham Chambers-Wall
(WashU GS)

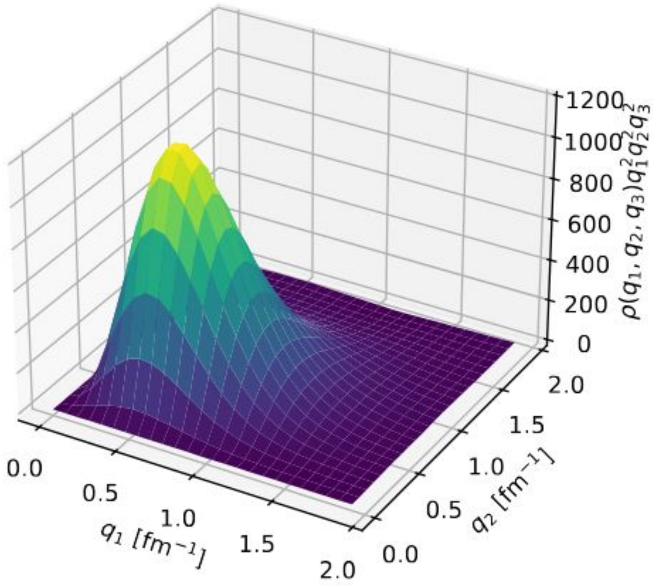
Garrett King
(LANL Director Fellow)

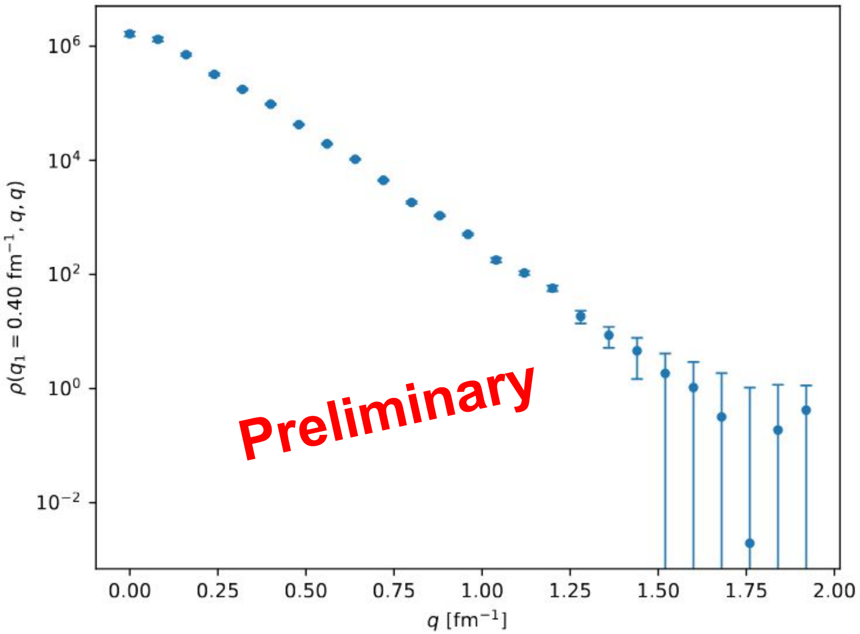
Lorenzo Andreoli
(ODU/JLab PD)

King *et al.* PRC 110 (2024) 5, 054325; *Ann.Rev.Nucl.Part.Sci. 74* (2024) 343
Chambers-Wall, Gnech, King *et al.* PRL 133 (2024) 21, 212501; PRC 110 (2024) 5, 054316
Andreoli *et al.* *PRC 110* (2024) 6, 064004

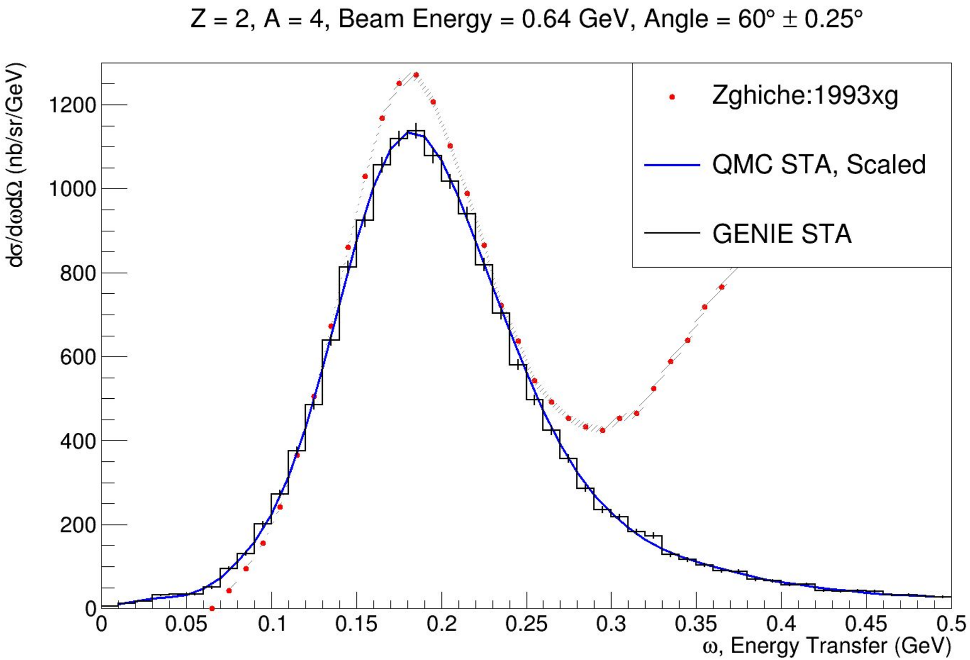
^{4}He Three-Body Momentum Distribution

$q_3 = 0.40$ [fm^{-1}]





GENIE validation using e-scattering



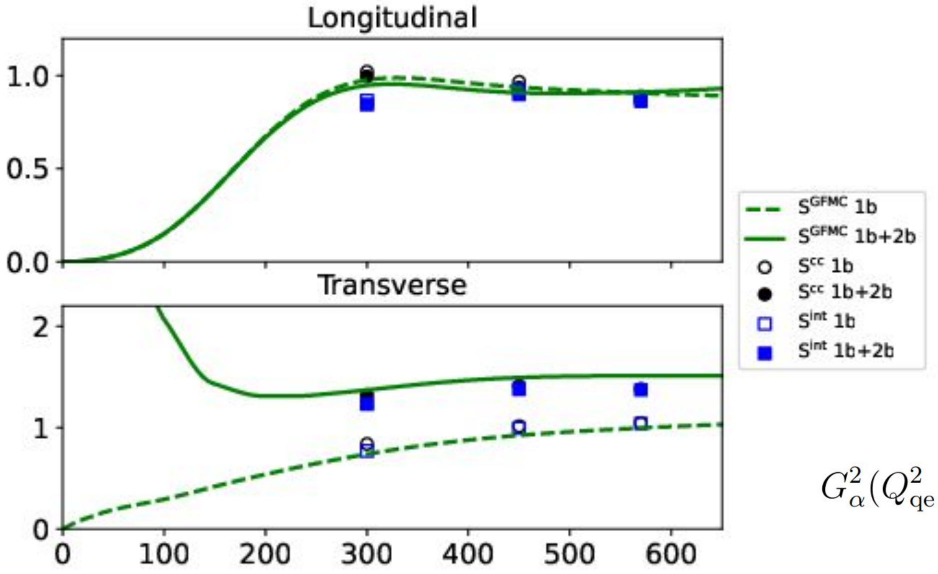
Barrow, Gardiner, Betancourt, SP *et al.* PRD103 (2021)

Ongoing work

- Implementation of moment-morphin interpolation techniques
- Implementations of response ***Densities*** in GENIE

$$\frac{d^2\sigma}{d\omega d\Omega} = \sigma_M \left[v_L R_L(\mathbf{q},\omega) + v_T R_T(\mathbf{q},\omega) \right]$$

^{12}C Sum Rules and Two-Nucleon Effects



Andreoli *et al.* PRC 110 (2024)

$$G_\alpha^2(Q_{\rm qe}^2)\, S_\alpha(q) = \int_{\omega_{\rm el}}^{\infty} d\omega\, R_\alpha(q,\omega) = \overline{\sum_{M_i}} \langle \Psi_i | O_\alpha^\dagger(\mathbf{q})\, O_\alpha(\mathbf{q}) | \Psi_i \rangle$$