GP Field Solver

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Grid and charge deposition

- Goal is to solve Poisson's equation

$$\nabla_{\perp}^{2}\phi(x,y) = -\frac{\rho(x,y)}{\varepsilon_0}$$

- Each longitudinal slice is treated independently in the (x,y) plane
- The transverse domain size [-X,X] x [-Y, Y] is defined by the grid
- The domain is divided into nx by ny cells
- Each cell stores the charge density from particles inside it
- Charge density computed from the particle count in each cell:

$$\rho_{ij} \approx \frac{n_{ij}e}{\Delta x\,\Delta y\,\Delta z}$$

Green's function solution

- The potential can be written as a convolution of rho with a kernel
- The kernel is the 2D Green's function of Poisson's equation

$$G(x, y) = \frac{1}{4\pi\varepsilon_0} \ln(x^2 + y^2)$$

- Each cell contributes to the potential everywhere on the grid
- Directly computing this sum is computationally expensive
- Potential as a convolution, and its discrete form:

$$\phi(x, y) = (\rho * G)(x, y) \qquad \phi(x, y) \approx \sum_{i,j} \rho_{ij} G(x - x_i, y - y_j)$$

FFT acceleration

- Direct convolution scales poorly with grid size
- The convolution theorem allows the calculation to be done in Fourier space

$$\mathcal{F}\{\phi\} = \mathcal{F}\{\rho\}\,\mathcal{F}\{G\}$$

- The charge density and Green's kernel are Fourier transformed
- Multiplication replaces the expensive convolution
- The potential is recovered with an inverse FFT

$$\rho \to \mathrm{FFT} \to \tilde{\rho} \to \times\tilde{G} \to \mathrm{IFFT} \to \phi$$

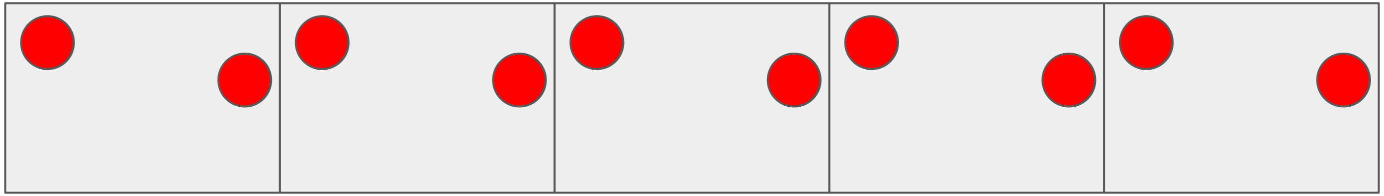
- Both beams are packed into one complex FFT to reduce runtime

$$\rho_{\mathrm{packed}} = \rho_1 + i\rho_2$$

Zero padding

- FFT convolution is periodic and causes wrap-around interactions
- Without correction, the left and right boundaries would interact
- The charge grid is embedded into a larger 2nx by 2ny padded grid
- Padded region contains 0 charge density
    - This separates the beam from its periodic images
- Only the central physical region is kept after the solve!

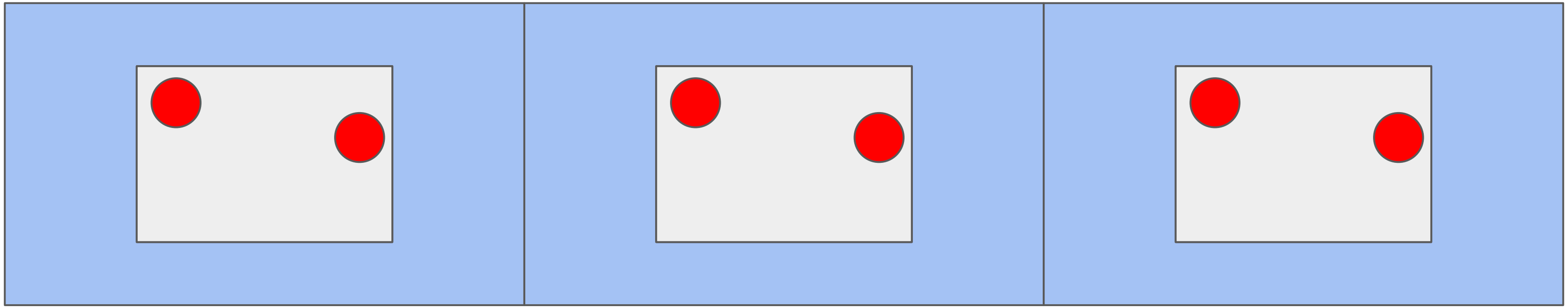
Zero padding



Unpadded:

- charges from neighboring tiles influence each other,
- boundary conditions are unphysical

Padded version fixes this:



Electric and magnetic fields

- The electric field is computed from the gradient of the potential
- Only transverse components are solved in the slice
- The beams are ultra-relativistic and travel along z
- The magnetic field is reconstructed from the electric field
- The resulting fields are purely transverse

$B_x = -E_y$ $\qquad B_y = E_x$ $\qquad B_z = 0$