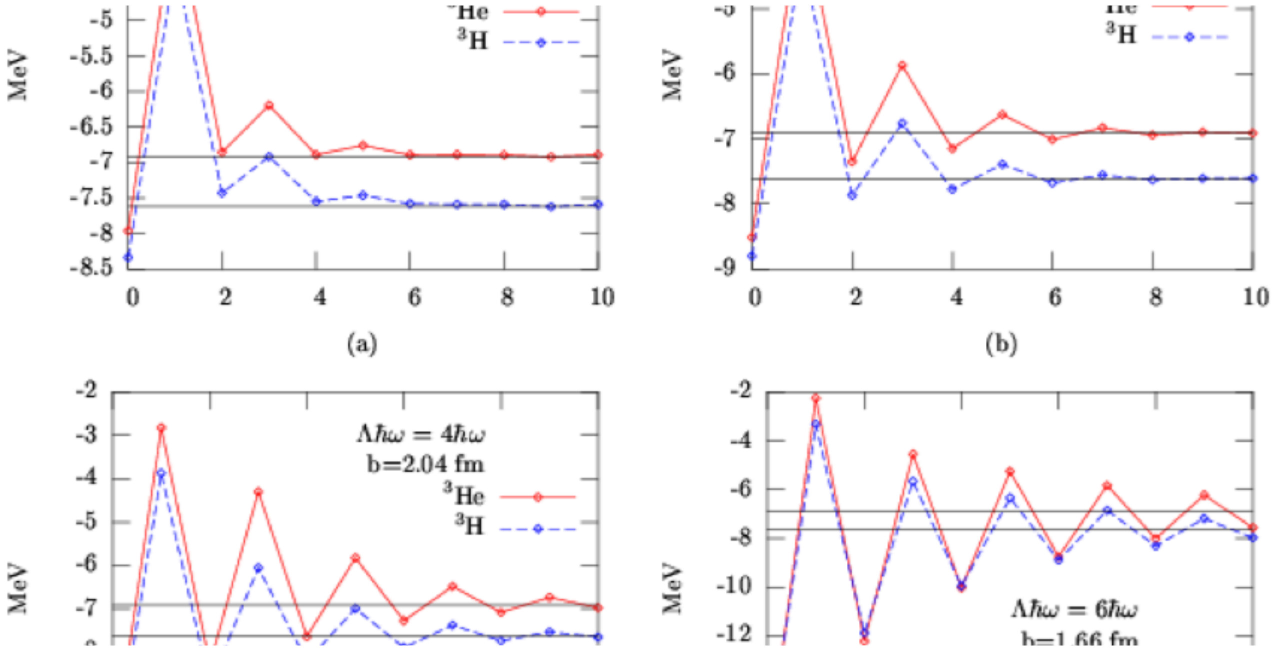

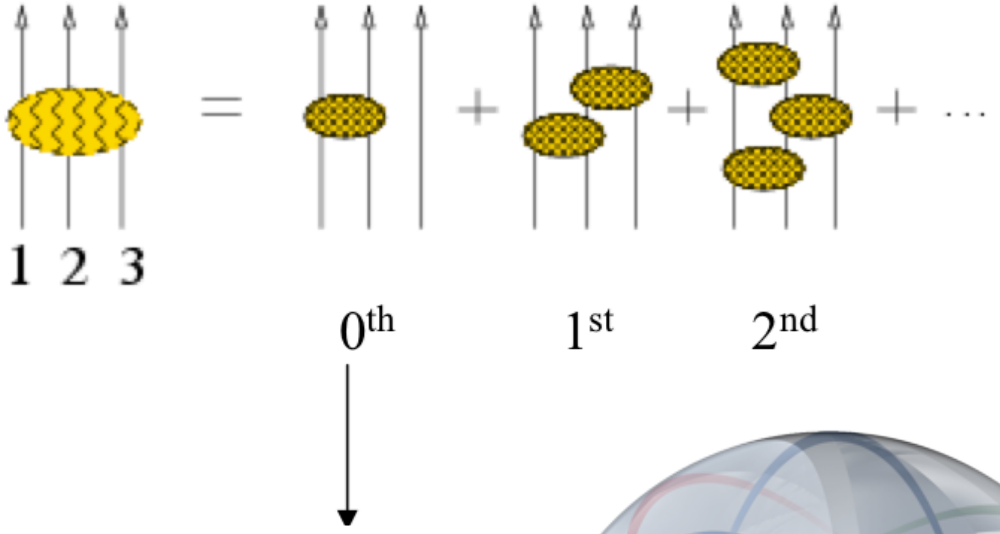

RE-VISITING THE BLOCH-HOROWITZ EQUATION

AND REMINISCING AT THE SAME TIME

THOMAS LUU (FZJ, UNI-BONN)

WICKFEST JAN. 7-8, 2020

BERKELEY

UNIVERSITÄT BONN

JÜLICH Forschungszentrum

WHO I AM

"No, I'm not a neutrino physics expert"

- Wick's student from 1999-2003 (@ UW)
- PHD thesis: "Effective interactions in an Oscillator basis" (see Ken's talk)

⋮

- FZJ/Uni-Bonn (2013-present)

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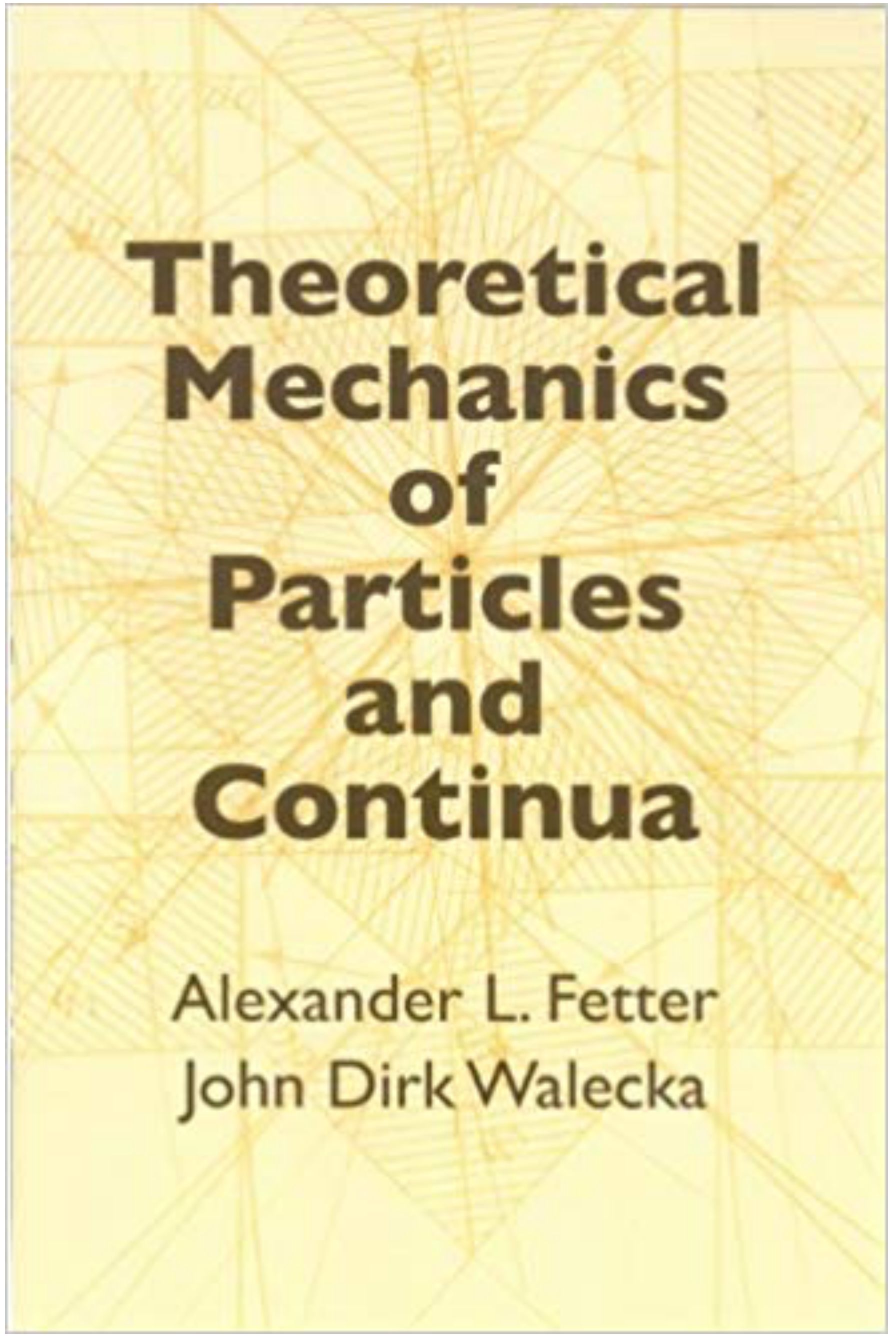
Let the reminiscing commence. . .

MY FIRST EXPOSURE TO WICK

Fall 1997

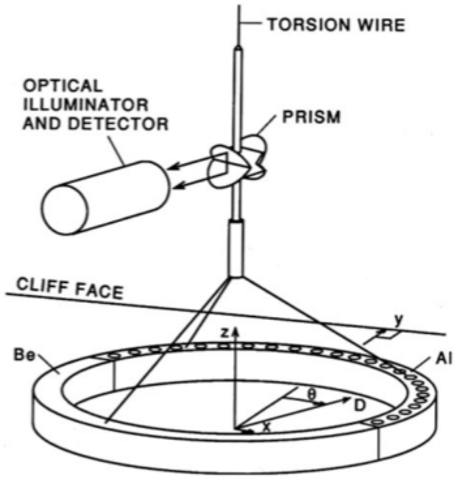
- Took course on *Classical Mechanics* given by Wick
- My observations
  - Wick assigned the homework problems
  - Wick wrote solutions to the homework problems himself—thought that was kind of cool
  - Wick's lecturing style resonated with me
- Gave series of lectures on ***Chaotic Dynamics***
  - My first real introduction to numerics

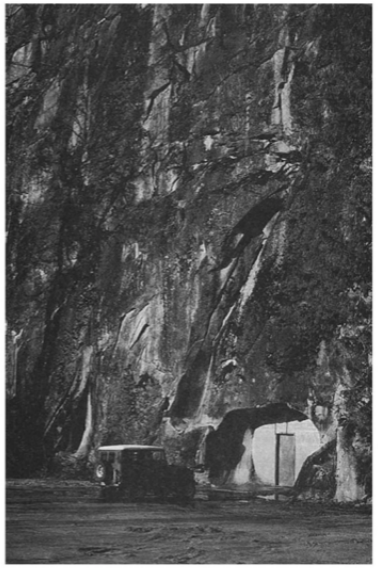


SEARCHING FOR A PH.D. ADVISOR

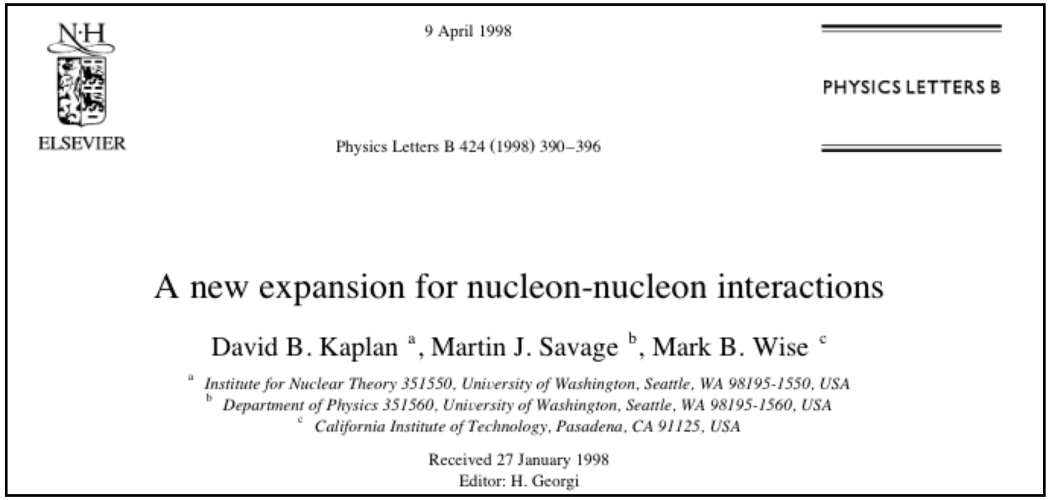
1998-1999

- Worked in P. Boynton's gravity lab
    - learned that I was not cutout to be an experimentalist
    - but got a lot of "hands-on" experience





- Did *Independent Research Project* related to EFT with M. Savage
    - "intense" experience. . .
    - . . . and a little "scary"



ELSEVIER
9 April 1998
PHYSICS LETTERS B
Physics Letters B 424 (1998) 390–396

A new expansion for nucleon-nucleon interactions

David B. Kaplan [a], Martin J. Savage [b], Mark B. Wise [c]
[a] Institute for Nuclear Theory 351550, University of Washington, Seattle, WA 98195-1550, USA
[b] Department of Physics 351560, University of Washington, Seattle, WA 98195-1560, USA
[c] California Institute of Technology, Pasadena, CA 91125, USA

Received 27 January 1998
Editor: H. Georgi

CONVERGING TO WICK AS A PH.D. ADVISOR

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- Proactive professors would recruit the best students

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  - Wick did NOT recruit me

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  - He recruited Marina Hruška
    - worked on project related to fractional QHE
    - numerical aspect

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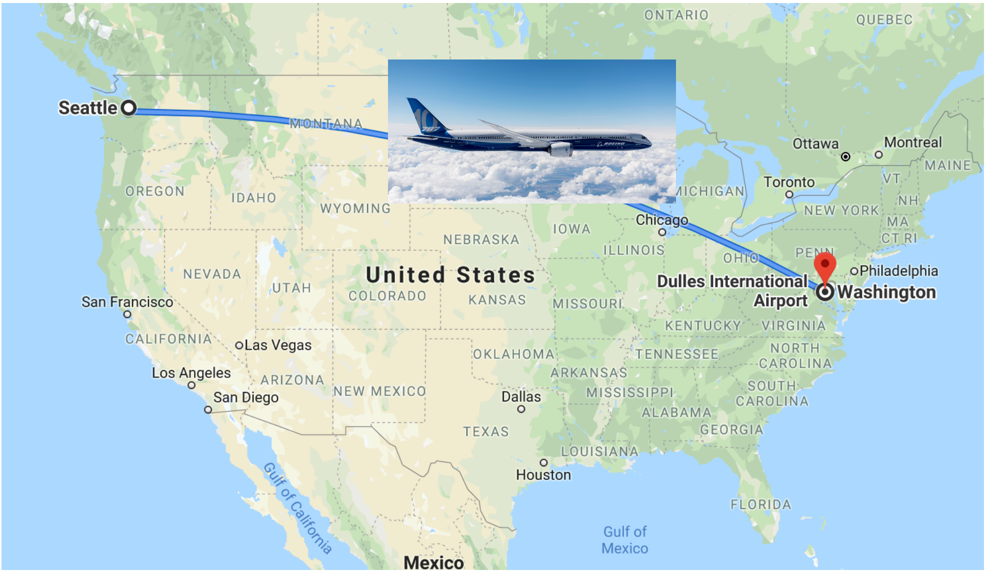
    - worked on project related to fractional QHE
    - numerical aspect
    - I was jealous

- But it’s all about timing
  - Wick’s prior student (Song) recently finished
  - I continued this work. . .

LIFE UNDER WICK'S TUTELAGE

Life was pretty good!



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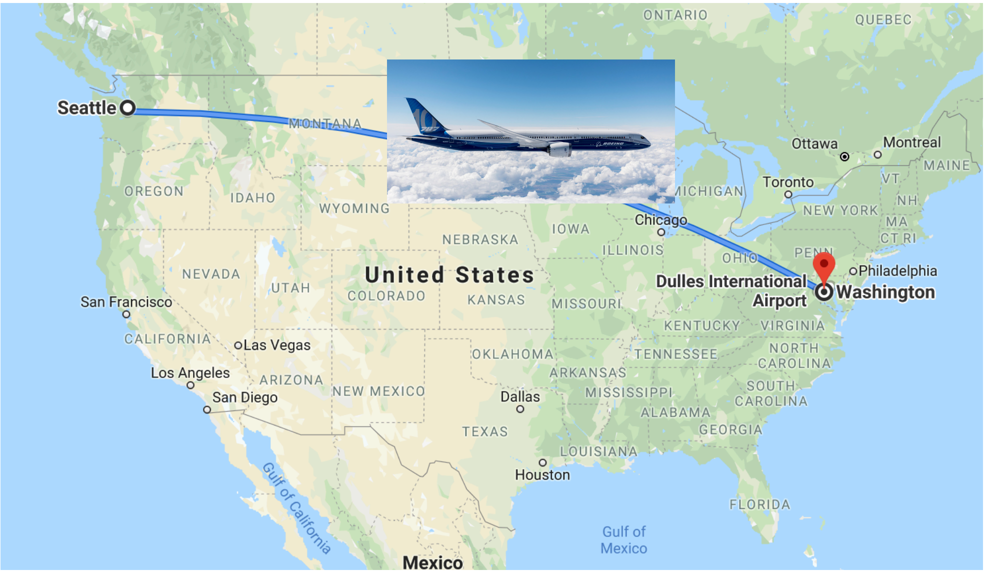

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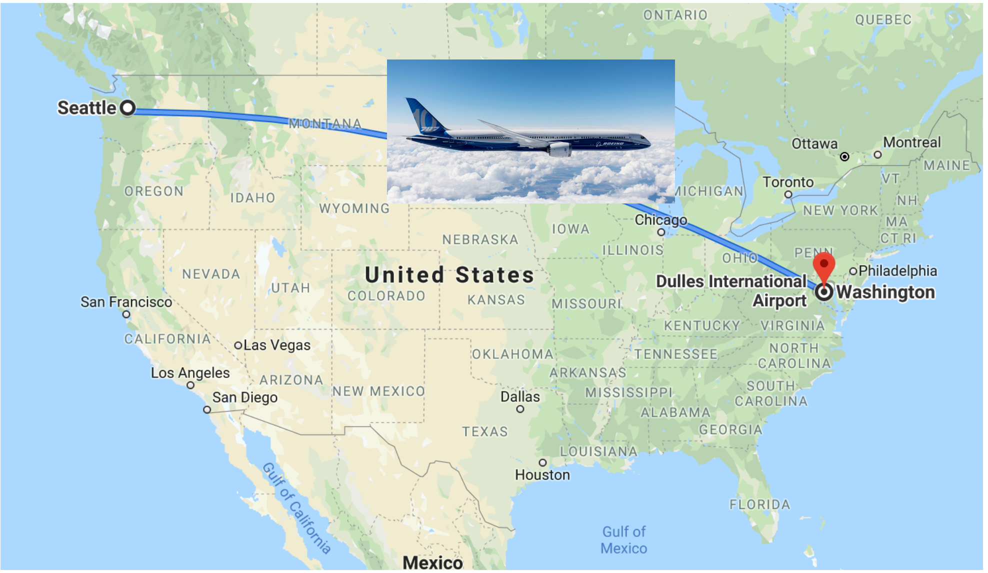
  - Constantly jet-setting to the east coast (and other places)
  - “when the cat’s away, the mice will play”



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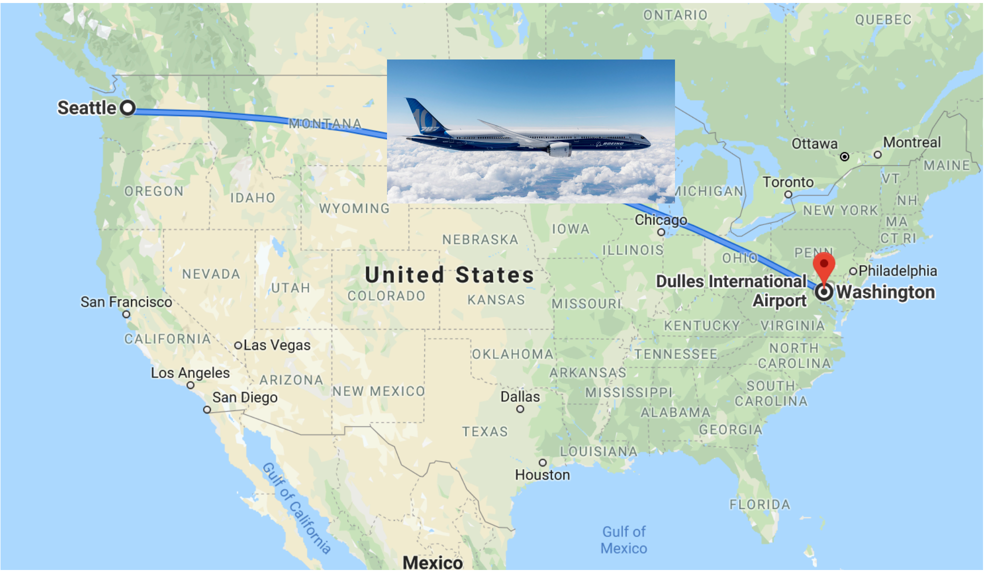
- Wick knew when to intervene
  - I was very "green behind the ears", but I got guidance when I needed it



"I GET MY BEST WORK DONE WHILE ON THE PLANE"

— Wick Haxton, ca. 2000

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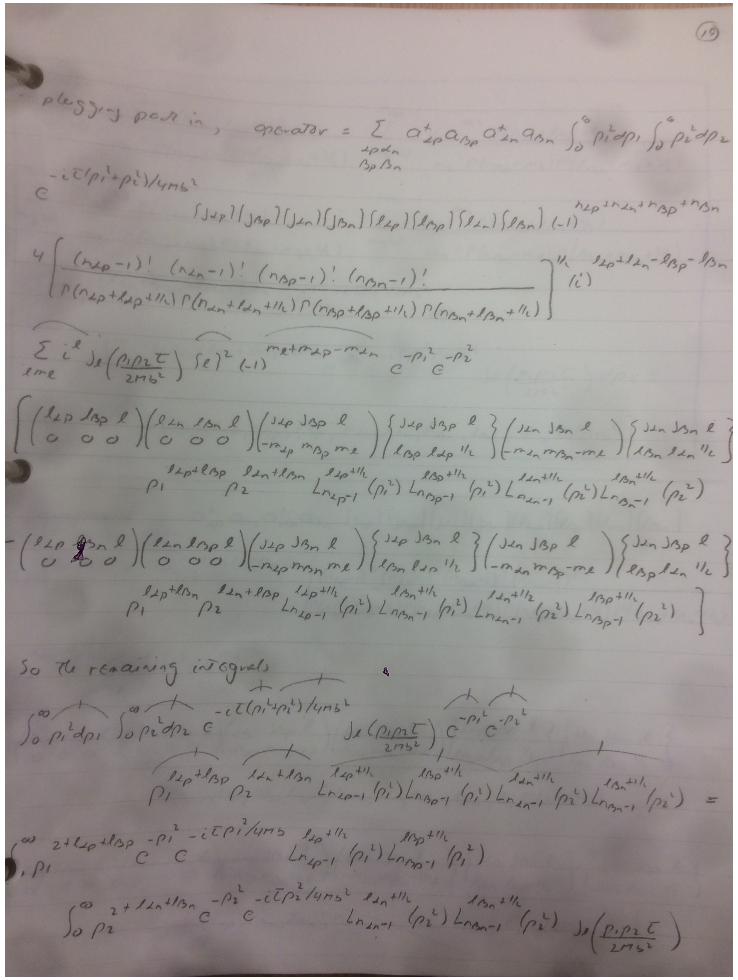

"Code this up"

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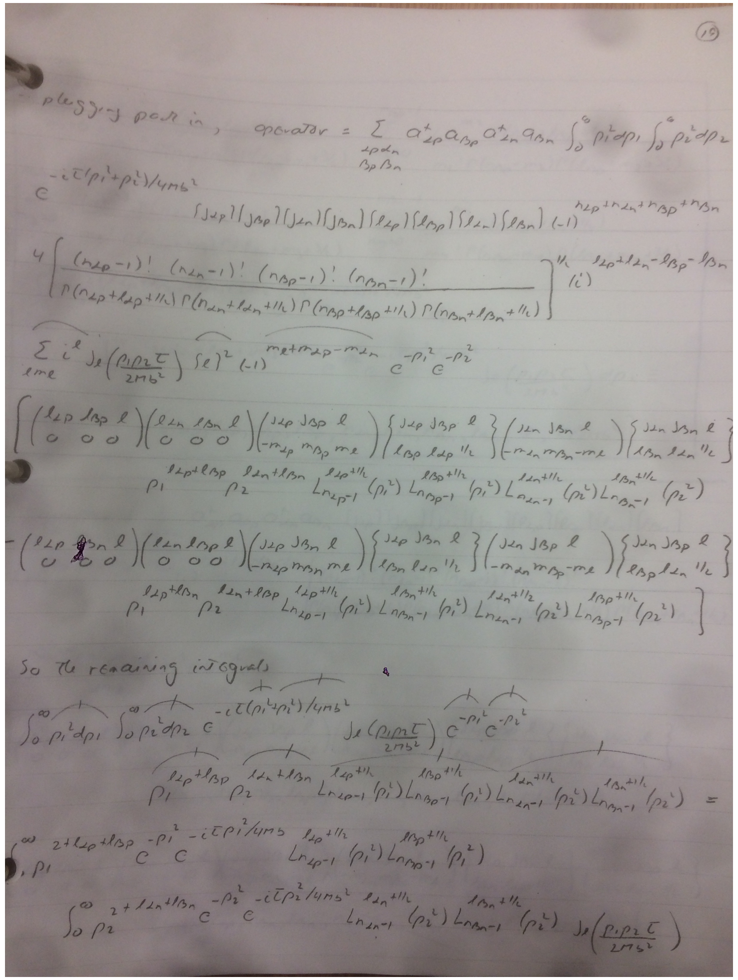

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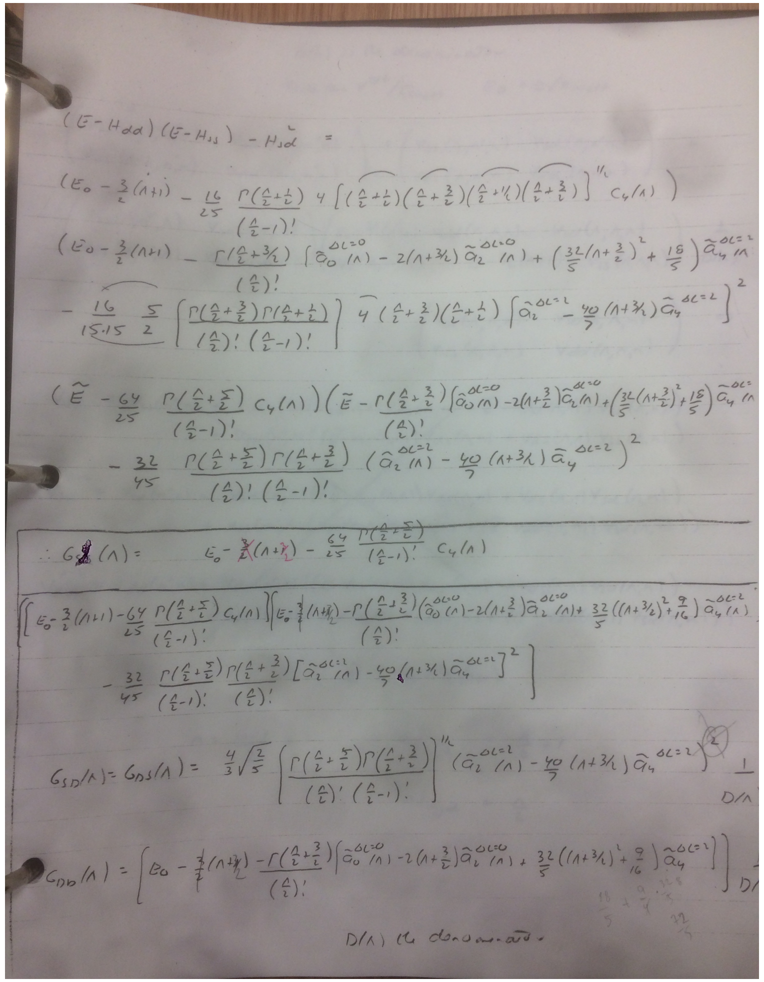

"Is that a typo?"

WICK, IF YOU'RE MISSING ANY OF YOUR NOTES. . .

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“WE FOLLOW OUR NOSES”

Numerics and theory complement each other

- Wick’s sense of smell was pretty good!

- My “love” of numerics really grew under Wick’s tutelage

“EXACTLY WHO IS *THEY?*”

Calculating the Deuteron BE to KeV precision

- Early on, were were able to calculate Deuteron BE to within 100 KeV

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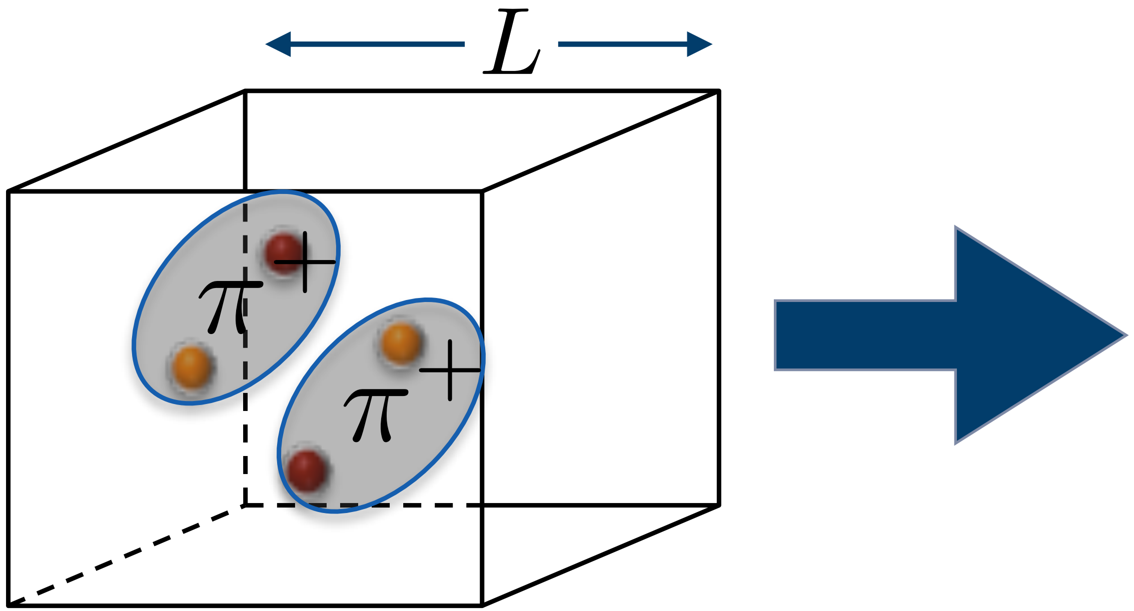
  - My (internal) response: “Am I ever getting my Ph.D.?”

LÜSCHER'S FORMULA

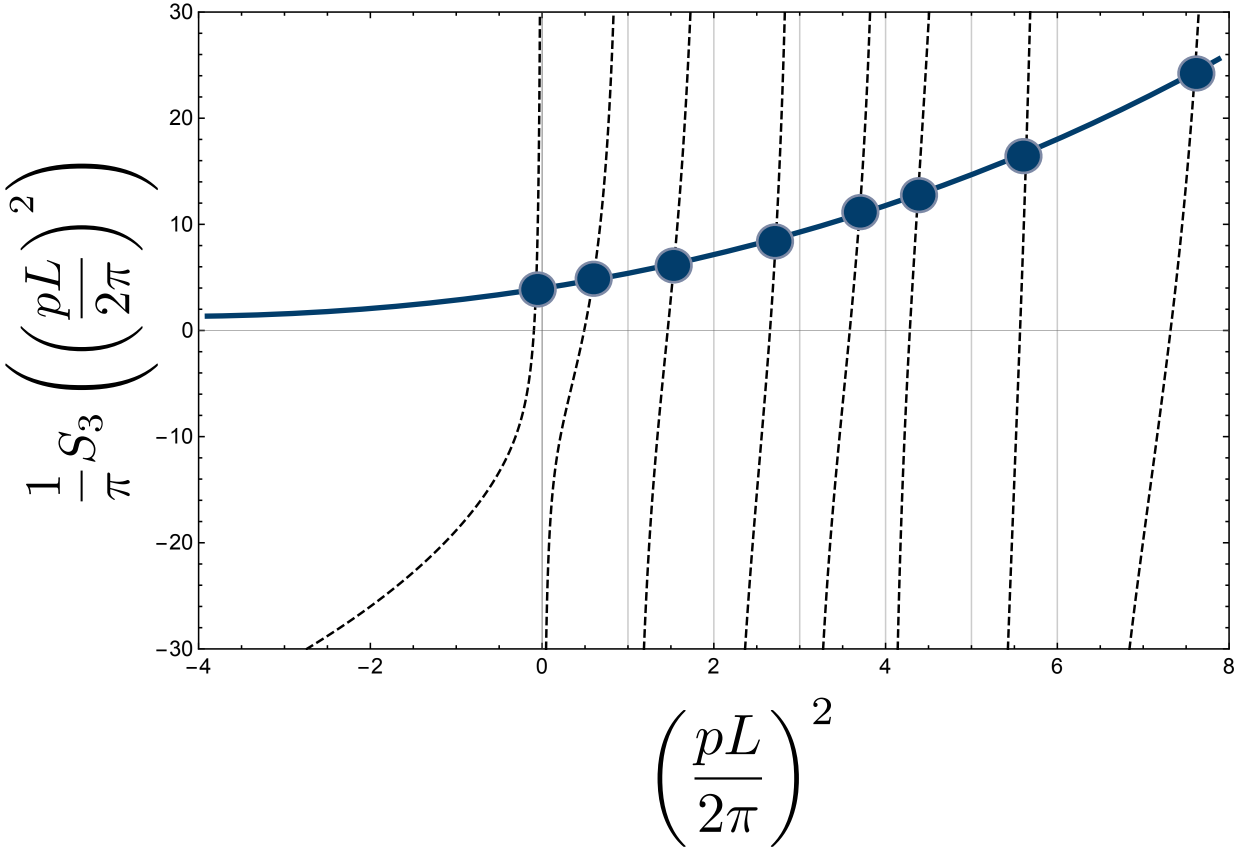
Re-visiting the Bloch-Horowitz equation

$$p L \cot \delta(p) = \lim_{\Lambda \to \infty} \frac{1}{\pi} \sum_{\vec{n}}^{\Lambda L / 2\pi} \frac{1}{\vec{n}^2 - \left(\frac{pL}{2\pi}\right)^2} - \frac{2\Lambda}{\pi} \equiv \frac{1}{\pi} S_3\left(\left(\frac{pL}{2\pi}\right)^2\right)$$

$$= -\frac{L}{a} + \frac{L}{2} r p^2 + \cdots$$ (effective range expansion)



finite volume eigenenergies $\left(\frac{pL}{2\pi}\right)^2$



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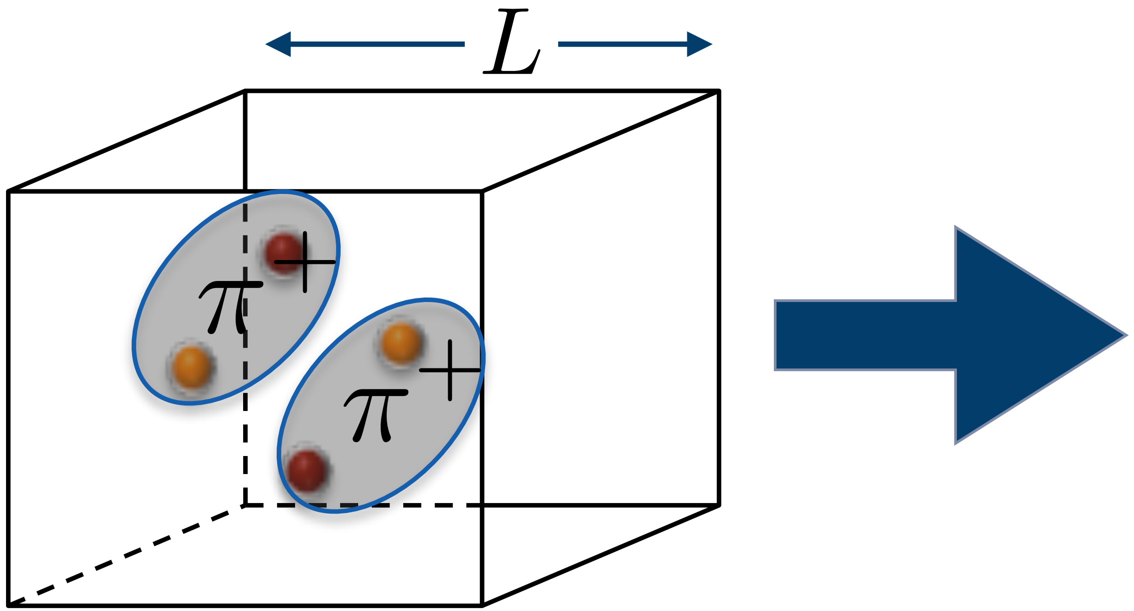

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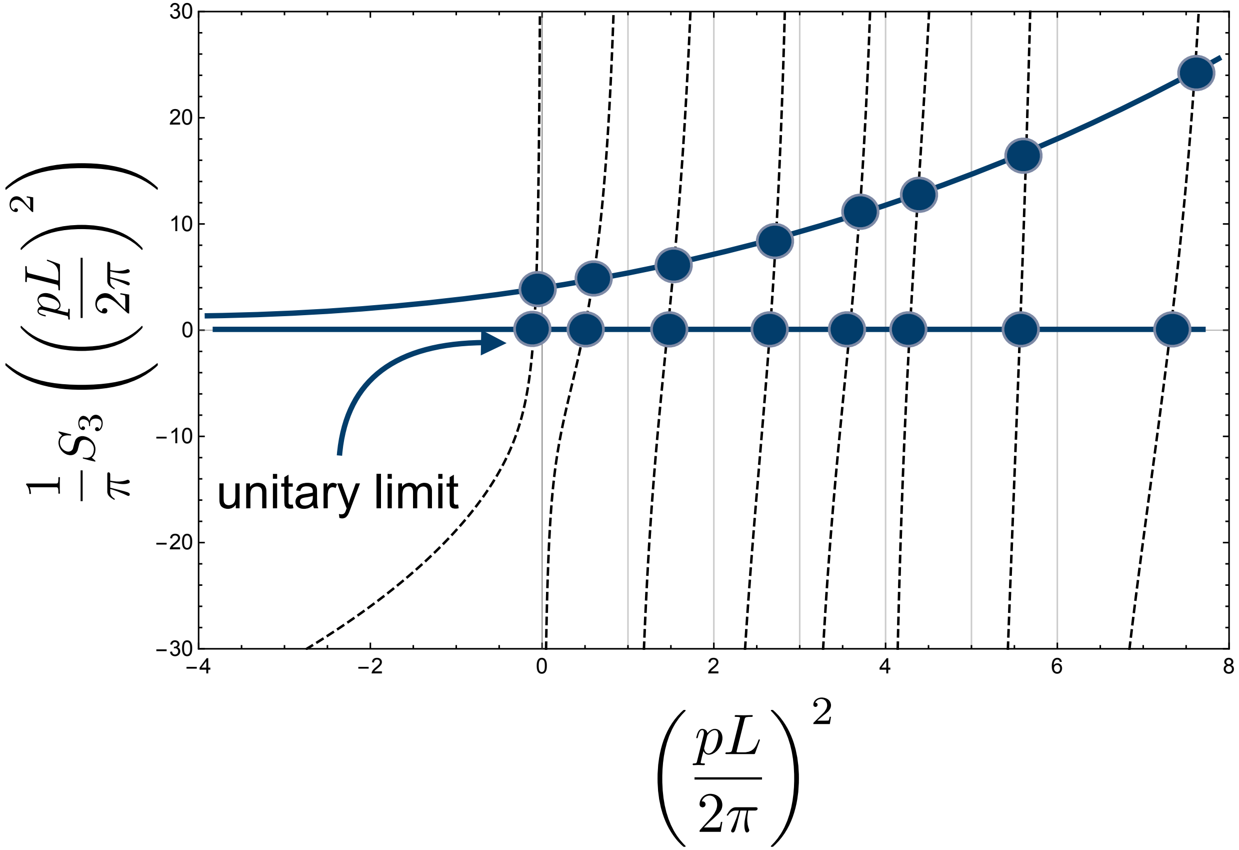

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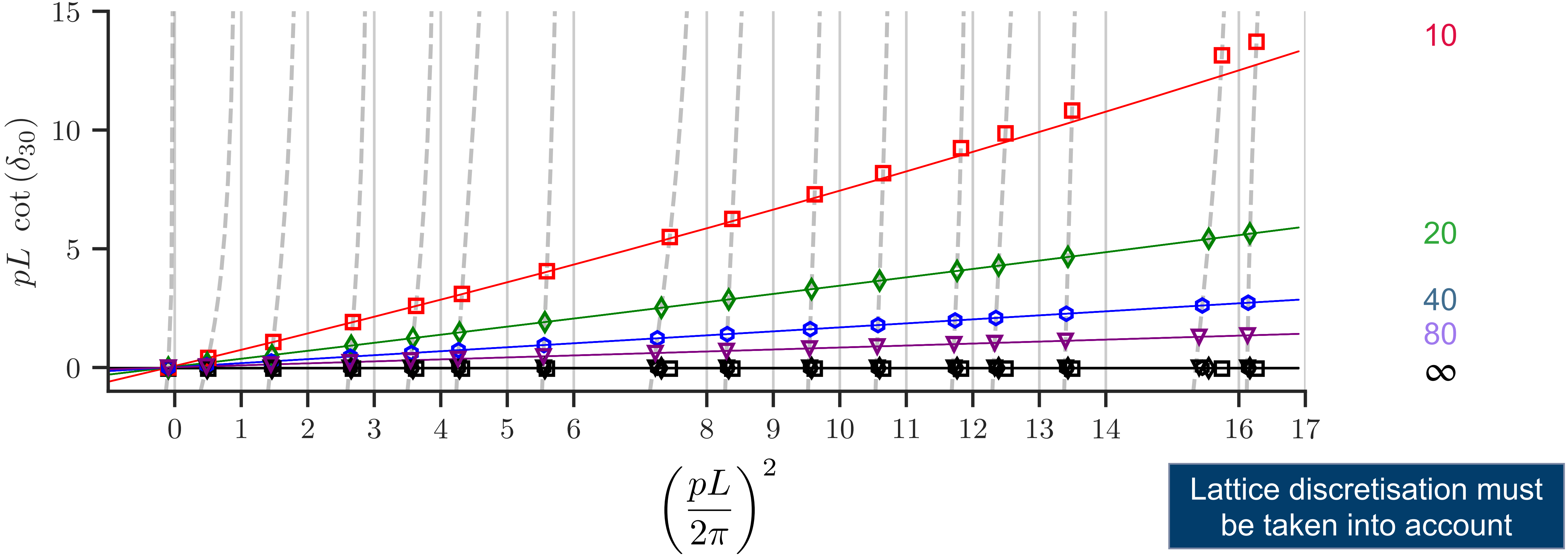
INDUCED MOMENTUM TERMS



Lattice discretisation must be taken into account

Seki & van Kolck, [nucl-th/0509094] Phys.Rev. **C73** (2006) 044006

Endres et al., [arXiv:1106.5725] Phys.Rev. **A84** (2011) 043644
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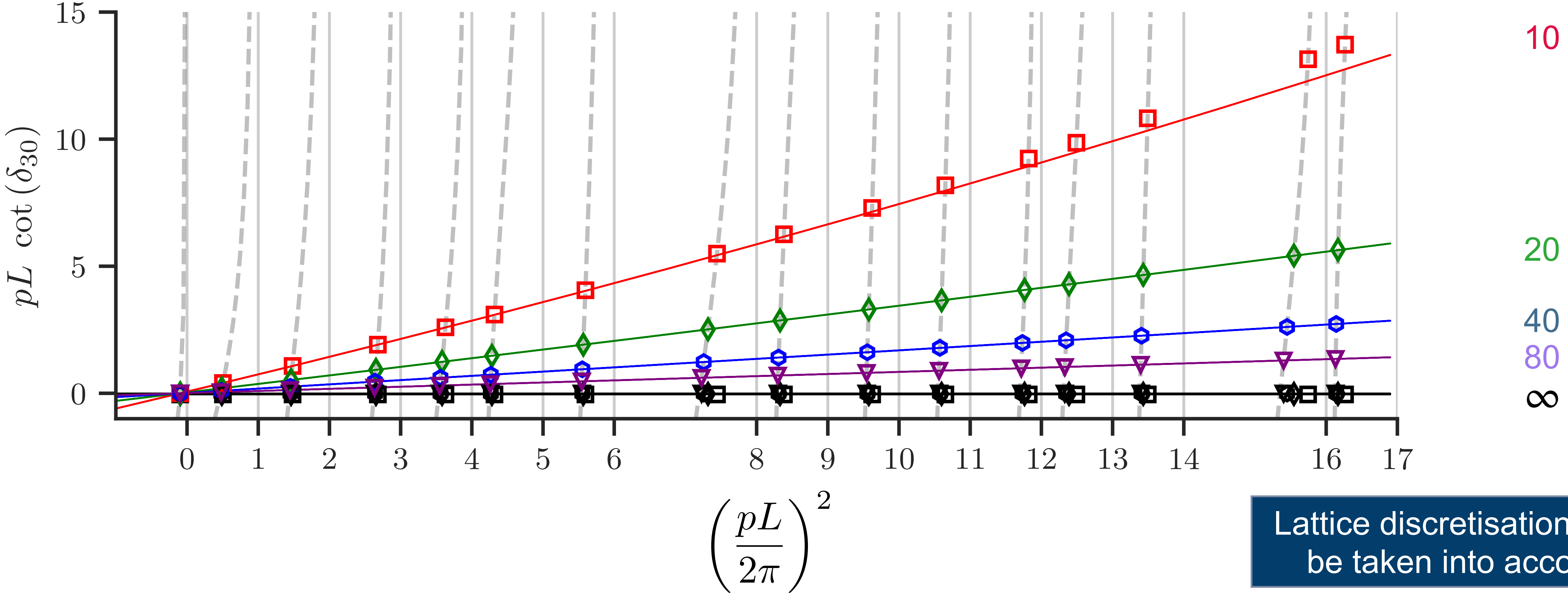

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} Tuned higher-order, momentum-dependent interactions to remove this behaviour

$$+\alpha_1 p^2 + \alpha_2 p^4 + \ldots$$

THE EFFECTIVE HAMILTONIAN

Bloch-Horowitz equation

Have $$H|\psi\rangle = E|\psi\rangle$$

Want $$H_{eff}|\psi_P\rangle = E|\psi_P\rangle$$

wave function projected into "included space"

same *E* as in the full space

Formally, the solution to H_{eff} is $$H_{eff} = H + HQ\frac{1}{E - QHQ}QH$$

operator only acts in P space!

energy-*dependent* equation

SPLITTING INTO P AND Q SPACES

Discretized lattice induces a cutoff in momentum modes: define this as the P space

$$\hat{1}_{cont.} = \frac{1}{L^3} \sum_{n=-\infty}^{\infty} |p_n\rangle\langle p_n| = \frac{1}{L^3} \sum_{n \in \mathrm{B.Z.}} |p_n\rangle\langle p_n| + \frac{1}{L^3} \sum_{n \notin \mathrm{B.Z.}} |p_n\rangle\langle p_n|$$
$$\equiv \hat{P} + \hat{Q}$$

definition of P and Q spaces

$$n \in \mathrm{B.\,Z.} \implies n_{x,y,z} \in \left(-\frac{N}{2}, \frac{N}{2}\right]$$

definition of Brillouin Zone
(P space)

$$|\psi\rangle = (P + Q)|\psi\rangle \equiv |\psi_P\rangle + |\psi_Q\rangle$$

projection of state into P and Q components

APPLYING THIS TO THE CONTACT INTERACTION

Unitary limit with a delta function

$$H|\psi\rangle = (T + V^\delta)|\psi\rangle = E|\psi\rangle$$

defining equation

$$V^\delta = \lim_{\Lambda\to\infty} C(\Lambda) \prod_{i=x,y,z} \Theta\left(\frac{\Lambda}{2} - |p_i|\right)$$

I put the cutoff into the definition of the potential

$$C(\Lambda) = -\frac{4\pi/m}{-\frac{1}{a} + \mathcal{L}_3^\square \frac{\Lambda}{4\pi^2}}$$

Yes, I used *m* instead of *μ*, so sue me!

SOLUTION IN CASE OF CONTACT INTERACTION

Not surprisingly, we can solve the delta-function case

The solution requires using properties like $[T,P]=[T,Q]=0$ and $T|n\rangle = |n\rangle \frac{\boldsymbol{p}_n^2}{m}$

no kinetic energy resummations needed!

$$H_{eff} = T + V^\delta + V^\delta Q \frac{1}{E - T - QV^\delta Q} QV^\delta$$

$\vdots$ → skipping lots of steps (e.g. geometric sum, etc.)

$$= T + \frac{1}{\frac{1}{C(\Lambda)} - I_1^Q(E)} \quad (= C(N,E))$$

$$\equiv T + V_{eff}^\delta(E)$$

energy-*dependent*!

$$I_1^Q(E) = \frac{1}{L^3} \sum_{n \notin \mathrm{B.Z.}} \frac{1}{E - \frac{\boldsymbol{p}_n^2}{m}}$$

$$= -\frac{m}{4\pi^2 L} \sum_{n \notin \mathrm{B.Z.}} \frac{1}{\boldsymbol{n}^2 - x}$$

$$= -\frac{m}{4\pi^2 L} \mathcal{I}_1^Q(x)$$

ASSUME ENERGIES $E = \frac{p^2}{m}$ ARE WELL WITHIN THE INCLUDED SPACE

$$\frac{1}{C(N,E)} = \frac{1}{C(\Lambda)} - I_1^Q(E) = \frac{1}{C(\Lambda)} + \frac{m}{4\pi^2 L} \sum_{n \notin \text{B.Z.}} \frac{1}{\boldsymbol{n}^2 - x}$$

$$x \equiv \left(\frac{pL}{2\pi}\right)^2$$

expand in small x

$$= \frac{1}{C(\Lambda)} + \frac{m}{4\pi^2 L} \sum_{n \notin \text{B.Z.}} \left(\frac{1}{\boldsymbol{n}^2} + \frac{x}{(\boldsymbol{n}^2)^2} + \frac{x^2}{(\boldsymbol{n}^2)^3}\right) + \mathcal{O}(x^3)$$

substitute expression for $C(\Lambda)$

$$= \frac{\frac{-1}{a_0} + \mathcal{L}_3^{\square} \frac{\Lambda}{4\pi^2}}{-4\pi/m} + \frac{m}{4\pi^2 L} \sum_{n \notin \text{B.Z.}} \left(\frac{1}{\boldsymbol{n}^2} + \frac{x}{(\boldsymbol{n}^2)^2} + \frac{x^2}{(\boldsymbol{n}^2)^3}\right) + \mathcal{O}(x^3)$$

collect x-independent terms

$$= \left(\frac{-m}{4\pi L}\right) \underbrace{\left(\frac{-L}{a_0} + \mathcal{L}_3^{\square} \frac{\Lambda L}{4\pi^2} - \frac{1}{\pi} \sum_{n \notin \text{B.Z.}} \frac{1}{\boldsymbol{n}^2}\right)} + \frac{m}{4\pi^2 L} \sum_{n \notin \text{B.Z.}} \left(\frac{x}{(\boldsymbol{n}^2)^2} + \frac{x^2}{(\boldsymbol{n}^2)^3}\right) + $$

This term is well behaved in the limit $\Lambda \to \infty$

So we have $$\frac{1}{C(N,E)} = \frac{1}{C(\Lambda)} - I_1^Q(E) \approx \underbrace{\left(\frac{-m}{4\pi L}\right)\left(\frac{-L}{a_0} + \frac{1}{\pi}\mathcal{L}_3^{\square}\frac{N}{2}\right)}_{\frac{1}{C(N,0)}} + \underbrace{\frac{m}{4\pi^2 L}\sum_{n\notin \text{B.Z.}}\left(\frac{x}{(\boldsymbol{n}^2)^2} + \frac{x^2}{(\boldsymbol{n}^2)^3}\right) + \mathcal{O}(x^3)}_{\text{momentum-induced terms}}$$

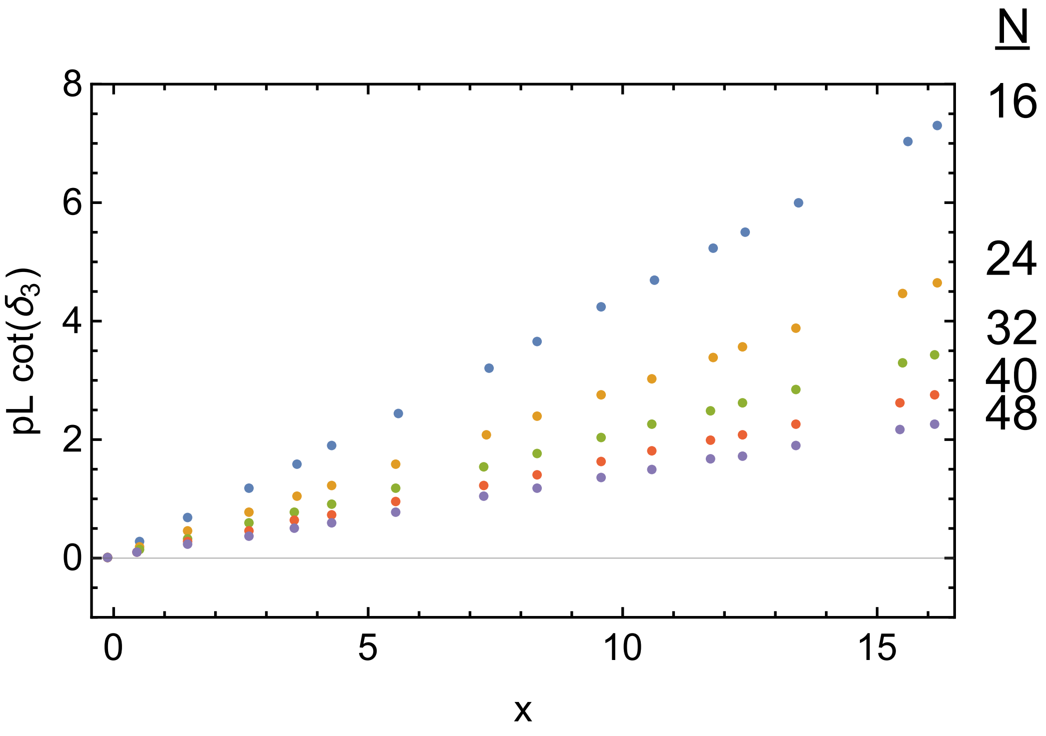
$$\implies \frac{-L}{a_0} - \frac{1}{\pi}\sum_{n\notin \text{B.Z.}}\left(\frac{x}{(\boldsymbol{n}^2)^2} + \frac{x^2}{(\boldsymbol{n}^2)^3}\right) + \mathcal{O}(x^3) = \frac{1}{\pi}S_3^{\boxplus}(x)$$

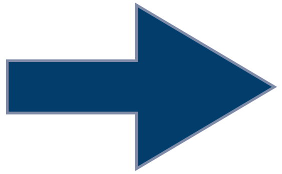
$$= \frac{-L}{a_0} - \frac{1}{\pi}\alpha_2(N)x + \alpha_3(N)x^2 + \mathcal{O}(x^3) = \frac{1}{\pi}S_3^{\boxplus}(x)$$

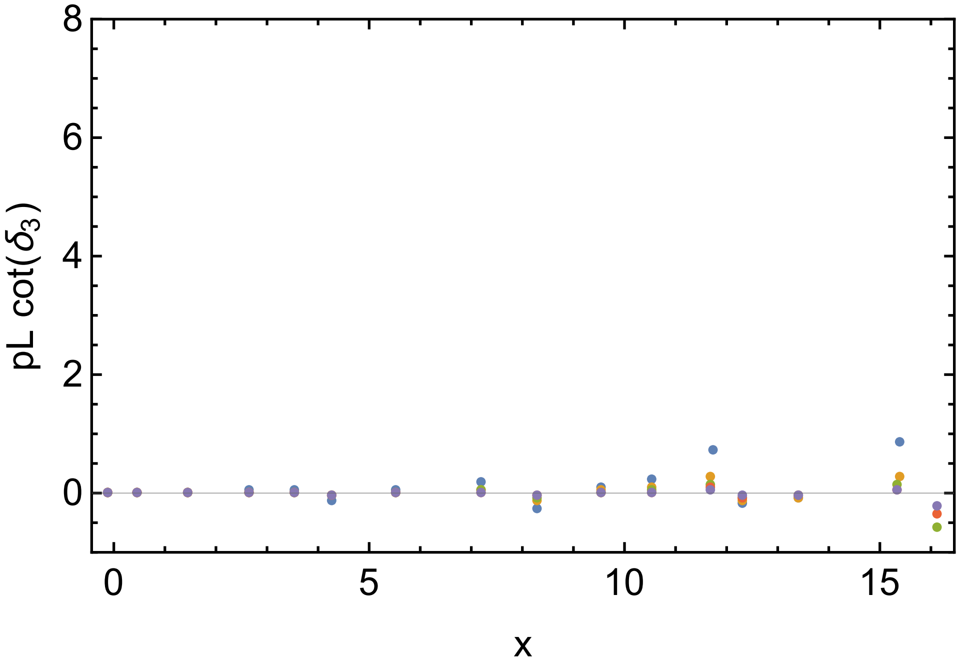
N	α_1	α_2	α_3
10	0.346 228 470 193 45	2.108 836 129 902 6	0.020 967 281 332 39
20	0.173 840 297 984 83	1.047 005 248 267 3	0.002 537 745 887 32
40	0.087 011 479 757 28	0.522 565 277 653 1	0.000 314 563 119 10
50	0.069 617 964 079 68	0.417 962 000 493 6	0.000 160 892 376 74
80	0.043 517 174 427 02	0.261 165 126 818 4	0.000 039 236 957 20
100	0.034 814 837 651 36	0.208 920 812 867 4	0.000 020 084 189 57

When we use an energy-*independent* coefficient *C(N,0)* in the P space, we are essentially throwing away the energy-dependent part of the effective interaction.

NOW WE CAN CORRECT THE PHASE SHIFTS







SO I COME BACK TO THE BLOCH-HOROWITZ EQUATION AFTER 17 YEARS. . .

- Discretized space can be represented at the P (included) space
- The induced momentum terms are simply the energy-*dependent* part of the effective interaction that has been omitted

- Quite fortunate to have Wick as my advisor
- It has been an honor! Here’s to another 70 years!