Entanglement and collective flavor oscillations in a dense neutrino gas

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The N3AS collaboration

- Network in Neutrinos, Nuclear Astrophysics, and Symmetries - funded by National Science Foundation (NSF) and Heising-Simons Foundation
- Multi-institutional network (3 centers + 8 sites) dedicated to recruiting and training postdocs, fostering collaborative efforts, and advancing research in the following areas:
    - Neutrino physics and astrophysics
    - Nucleosynthesis
    - Dense matter
    - Dark matter
- 8 postdocs currently supported
- `https://n3as.wordpress.com/`

Today's talk

- Introduction to neutrino oscillations (effective one-particle description)
- Neutrino many-body Hamiltonian and its symmetries, eigenvalues, and eigenstates
- Using eigenvalues and eigenstates to study adiabatic flavor evolution of a neutrino many-body system
- Measures to quantify entanglement in the system
- Comparison between flavor evolution in the many-body approach and in the mean-field description

References

- Michael J. Cervia, Amol V. Patwardhan, A. B. Balantekin, S. N. Coppersmith, and Calvin W. Johnson
  Phys. Rev. D 100, 083001 (2019), arXiv:1908.03511

- Amol V. Patwardhan, Michael J. Cervia, and A. B. Balantekin
  Phys. Rev. D 99, 123013 (2019), arXiv:1905.04386

- Michael J. Cervia, Amol V. Patwardhan, and A. B. Balantekin
  IJMPE 28 (2019) 1950032, arXiv:1905.00082

Additional reading

- Ermal Rrapaj
  arXiv:1905.13335
- Savas Birol, Y. Pehlivan, A. B. Balantekin, and T. Kajino
  Phys. Rev. D 98, 083002 (2018), arXiv:1805.11767
- Y. Pehlivan, A. B. Balantekin, Toshitaka Kajino, and Takashi Yoshida
  Phys. Rev. D 84, 065008 (2011), arXiv:1105.1182
- Alexandre Faribault, Omar El Araby, Christoph Sträter, and Vladimir Gritsev
  Phys. Rev. B 83, 235124 (2011), arXiv:1103.0472
- Pieter W. Claeys
  arxiv:1809.04447 (PhD thesis)

Outline

1. Neutrino oscillations: effective one-particle description
2. Many-body treatment of neutrino oscillations
3. Adiabatic evolution and entanglement measures
4. Comparison with mean-field calculations

Neutrino oscillations (in vacuum)

- Neutrino weak-interaction (flavor) eigenstates not aligned with propagation (energy/mass) eigenstates

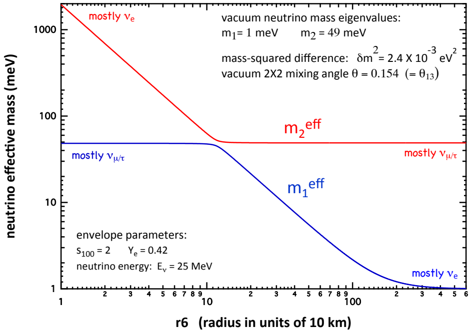
$$|\nu_e\rangle = \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle$$
$$|\nu_x\rangle = -\sin\theta\,|\nu_1\rangle + \cos\theta\,|\nu_2\rangle$$

- As neutrinos propagate, mass eigenstates gather quantum mechanical phase at different rates, leading to oscillations

$$P_{ex} = \sin^2 2\theta \sin^2\left(\frac{\Delta m^2 L}{4E_\nu}\right)$$

Neutrino flavor evolution: matter effects

Matter backgrounds (electrons, nucleons, etc.) modify flavor evolution: "effective mass" through neutrino forward scattering. Mass level crossing $H_{\nu_e\nu_e} = H_{\nu_x\nu_x} \implies$ MSW resonance



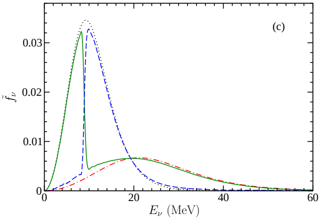
Wolfenstein (1978, '79)
Mikheyev & Smirnov (1985)
Bethe (1986)
Haxton (1986)
Parke (1986)
and so on ...

Neutrino-neutrino forward scattering

- Nonlinearity: Hamiltonian driving the flavor evolution depends on the flavor composition

$$H_{\nu\nu} = \sqrt{2}G_F \sum_{\alpha} \left[\int_{\nu} dn_{\nu,\alpha}\, \rho_{\nu,\alpha}(\boldsymbol{p}')(1 - \hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{p}}') - \int_{\bar{\nu}} dn_{\bar{\nu},\alpha}\, \rho_{\bar{\nu},\alpha}(\boldsymbol{p}')(1 - \hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{p}}') \right]$$

- Important at large neutrino number densities/fluxes: e.g., supernovae, compact object mergers, early universe



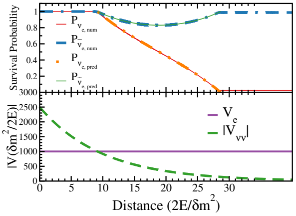


Figure: Left: spectral swap (Duan *et al.*, 2006). Right: matter-neutrino resonance (Malkus *et al.*, 2014).

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Neutrino oscillations: flavor/mass isospin operators

- Denote Fermionic operators for neutrino flavor/mass states as $a_\alpha(\mathbf{p})$, $a_j(\mathbf{p})$, where $\alpha = e, x$, and $j = 1, 2$

$$\begin{aligned} a_e(\mathbf{p}) &= \cos\theta\, a_1(\mathbf{p}) + \sin\theta\, a_2(\mathbf{p}) \\ a_x(\mathbf{p}) &= -\sin\theta\, a_1(\mathbf{p}) + \cos\theta\, a_2(\mathbf{p}) \end{aligned}$$

- Introduce the mass-basis isospin operators

$$J^+_{\mathbf{p}} = a_1^\dagger(\mathbf{p}) a_2(\mathbf{p}) \ , \qquad J^-_{\mathbf{p}} = a_2^\dagger(\mathbf{p}) a_1(\mathbf{p}) \ ,$$
$$J^z_{\mathbf{p}} = \frac{1}{2}\left(a_1^\dagger(\mathbf{p}) a_1(\mathbf{p}) - a_2^\dagger(\mathbf{p}) a_2(\mathbf{p})\right) \ ,$$

which obey the usual $SU(2)$ commutation relations

$$[J^+_{\mathbf{p}}, J^-_{\mathbf{q}}] = 2\delta_{\mathbf{pq}} J^z_{\mathbf{p}} \ , \qquad [J^z_{\mathbf{p}}, J^\pm_{\mathbf{q}}] = \pm\delta_{\mathbf{pq}} J^\pm_{\mathbf{p}} .$$

Neutrino oscillations: many-body Hamiltonian

- Vacuum oscillations:

$$H_{\text{vac}} = \sum_{\mathbf{p}} \left(\frac{m_1^2}{2p} a_1^\dagger(\mathbf{p}) a_1(\mathbf{p}) + \frac{m_2^2}{2p} a_2^\dagger(\mathbf{p}) a_2(\mathbf{p}) \right)$$
$$= \sum_{\omega} \omega \vec{B} \cdot \vec{J}_\omega \ ,$$

where $\omega = \frac{\delta m^2}{2|\mathbf{p}|}$, $\quad \vec{J}_\omega = \sum_{|\mathbf{p}| = \frac{\delta m^2}{2\omega}} \vec{J}_{\mathbf{p}}$, and

$\vec{B} = (0, 0, -1)_{\text{mass}} = (\sin 2\theta, 0, -\cos 2\theta)_{\text{flavor}}$.

- Neutrino-neutrino interactions

$$H_{\nu\nu} = \frac{\sqrt{2} G_F}{V} \sum_{\mathbf{p},\mathbf{q}} \left(1 - \cos \vartheta_{\mathbf{pq}}\right) \vec{J}_{\mathbf{p}} \cdot \vec{J}_{\mathbf{q}} \ .$$

Neutrino Hamiltonian: single-angle approximation

- Suitable averaging over the angle $\vartheta_{\mathbf{pq}}$ to simplify the problem

$$H_{\nu\nu} \approx \frac{\sqrt{2}G_F}{V}\langle(1-\cos\vartheta_{\mathbf{pq}})\rangle \vec{J}\cdot\vec{J} \qquad \text{(includes } J_p^2 \text{ terms)}$$

$$\equiv \mu(r)\vec{J}\cdot\vec{J}, \qquad \text{where } \vec{J} = \sum_{\omega} \vec{J}_\omega$$

- Many-body neutrino Hamiltonian with vacuum and ν–ν interactions (two-flavor, single-angle):

$$H_\nu = \sum_{p=1}^{M} \omega_p \vec{B}\cdot\vec{J}_p + \mu(r)\vec{J}\cdot\vec{J},$$

where p is an index for the ωs in the system, M in number

Mean-field (random phase) approximation

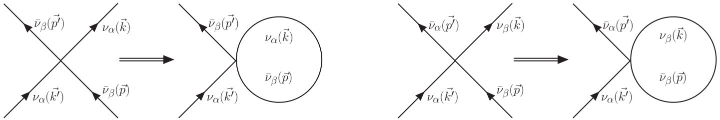

Figure: Volpe *et al.*, 2013

- In an effective one-particle approximation, a single neutrino is described as interacting with an average potential created by all other particles in the medium (including neutrinos)
- Operator product $\mathcal{O}_1\mathcal{O}_2$ approximated as

$$\mathcal{O}_1\mathcal{O}_2 \sim \mathcal{O}_1\langle\mathcal{O}_2\rangle + \langle\mathcal{O}_1\rangle\mathcal{O}_2 - \langle\mathcal{O}_1\rangle\langle\mathcal{O}_2\rangle.$$

Above expectation values are calculated w.r.t state $|\Psi\rangle$ which satisfies $\langle\mathcal{O}_1\mathcal{O}_2\rangle = \langle\mathcal{O}_1\rangle\langle\mathcal{O}_2\rangle$

Mean-field (random phase) approximation

- This method yields the effective one-particle neutrino Hamiltonian

$$H \sim H^{\text{RPA}} = \sum_{\omega} \omega \vec{B} \cdot \vec{J}_{\omega} + \mu \vec{P} \cdot \vec{J} \,,$$

where $\vec{P}_{\omega} = 2\langle \vec{J}_{\omega} \rangle$ is the "Polarization vector", and $\vec{P} = \sum_{\omega} \vec{P}_{\omega}$

- The self-consistency requirement of the mean-field approach then implies that $\vec{P}_{\omega}$ must satisfy

$$\frac{d}{dt}\vec{P}_{\omega} = (\omega \vec{B} + \mu \vec{P}) \times \vec{P}_{\omega}$$

Neutrino Hamiltonian: symmetries and invariants

- There exist N mutually commuting operators h_p:

$$h_p = -J_p^z + 2\mu \sum_{\substack{q=1 \\ q\neq p}}^{N} \frac{\vec{J}_p \cdot \vec{J}_q}{\omega_p - \omega_q} \qquad \text{(“Gaudin magnets”)}$$

- $[h_p, h_q] = 0$ and $H_\nu = \sum_p \omega_p h_p$, so that $[H, h_p] = 0$
- In addition, $J^z = \sum_p J_p^z$ also commutes with the Hamiltonian $\implies n_{\nu_1} - n_{\nu_2}$ is a conserved quantity
- Hamiltonian exhibits numerous energy level crossings— the invariants can be used to show the validity of the adiabatic approximation if μ varies smoothly

Neutrino Hamiltonian: eigenvalues and eigenstates

- Eigenvalues and eigenstates obtained using procedure derived from Richardson-Gaudin diagonalization (a.k.a. "Bethe-Ansatz" method) — Patwardhan et al., Phys. Rev. D 99, 123013 (2019)

- For a system where $j_p = 1/2 \; \forall p$, the eigenproblem can be mapped onto a system of coupled quadratic equations:

$$\tilde{\Lambda}_q^2 + \tilde{\Lambda}_q = \mu \sum_{\substack{p=1 \\ p\neq q}}^{N} \frac{\tilde{\Lambda}_q - \tilde{\Lambda}_p}{\omega_q - \omega_p}$$

  $\tilde{\Lambda}_p$ are related to eigenvalues of the invariants h_p. Bethe-Ansatz equations equivalent to polynomial relations between invariants h_p (Cervia et al., arXiv:1905.00082)

- Trivial solution for $\mu = 0$: each $\tilde{\Lambda}_q = 0$ or -1. Numerical solutions for $\mu > 0$ using homotopy continuation + Newton-Raphson

Neutrino Hamiltonian: eigenvalues and eigenstates

- In terms of the parameters $\Lambda_q = \tilde{\Lambda}_q/\mu$, the eigenvalues are given by

$$E(\Lambda_1, \ldots, \Lambda_N) = -\sum_p \frac{\omega_p}{2} + \mu \frac{N}{2}\left(\frac{N}{2}+1\right) - \mu \sum_p \omega_p \Lambda_p .$$

- Eigenstates are given by $e_\kappa \left|\nu_1, \ldots, \nu_1\right\rangle$, where the operator e_κ is the κ-th elementary symmetric polynomial of the Gaudin lowering operators. Can be calculated by recursively applying the following identities, for $k = 1, \ldots, \kappa$.

$$P_f(\Lambda_1, \ldots, \Lambda_N) = \sum_{p_1=1}^{M} \cdots \sum_{p_f=1}^{M} J^-_{p_1} \cdots J^-_{p_f} \sum_{m=1}^{f} \Lambda_{p_m} \prod_{\substack{l=1 \\ l\neq m}}^{f} \frac{1}{\omega_{p_l} - \omega_{p_m}} .$$

$$e_k(\Lambda_1, \ldots, \Lambda_N) = \frac{1}{k}\sum_{i=1}^{k} (-1)^{i-1} e_{k-i}\, P_i ,$$

Outline

Adiabatic evolution of a many-body neutrino system

- Eigenvalues and eigenvectors facilitate calculating the adiabatic evolution of the many-body neutrino system, starting from any given initial condition, as μ is varied

- Consider an initial many-body state, $|\Psi_0\rangle \equiv |\Psi(\mu_0)\rangle$
  - Example: in the (two-)flavor-basis, $|\nu_e \nu_x \nu_e \nu_e\rangle$

- May be decomposed into the basis of energy eigenstates: $|\Psi(\mu_0)\rangle = \sum_n c_n |e_n(\mu_0)\rangle$

- If μ were to change sufficiently slowly then the system adiabatically evolves into

$$|\Psi(\mu)\rangle \simeq \sum_n c_n e^{-i \int_{\mu_0}^{\mu} \frac{E_n(\mu')}{d\mu'/dt} d\mu'} |e_n(\mu)\rangle$$

Quantum entanglement in many-body neutrino systems

- In general, for $\mu > 0$, the eigenstates of the Hamiltonian are not factorizable into tensor products of individual neutrino states, and may therefore be described as entangled

- A system may initially start in a pure state—which happens to be a particular superposition of energy eigenstates. However, as the coefficients describing the superposition change with time (as do the eigenstates themselves), the system can become entangled. This is a feature unique to many-body systems, and cannot be observed in mean-field calculations

- Such entanglement may be quantified in terms of measures such as entropy of entanglement, length of individual neutrino polarization vectors, etc.

Summary of entanglement measures

Density Matrix, Polarization Vector, & Entanglement Entropy

Consider a pure, many-body neutrino state $\rho = |\Psi\rangle\langle\Psi|$.
Single-neutrino reduced density matrix: $\rho_q \equiv \mathrm{Tr}_{1,\ldots,\widehat{q},\ldots,N}[\rho]$, given by ($\widehat{\ }$ denotes exclusion)

$$\rho_q = \sum_{i_1,\ldots,\widehat{i_q},\ldots,i_N=1}^{2} \langle \nu_{i_1}\ldots\widehat{\nu_{i_q}}\ldots\nu_{i_N}|\rho|\nu_{i_1}\ldots\widehat{\nu_{i_q}}\ldots\nu_{i_N}\rangle\,,$$

- $S(\omega_q)$, Entropy of entanglement between neutrino q and rest:

$$S(\omega_q) = -\mathrm{Tr}[\rho_q \log \rho_q]$$

- "Polarization vector" of neutrino q, $\vec{P}(\omega_q) = 2\,\langle \vec{J}_q\rangle$, related to the reduced density matrix as:

$$\rho_q = \frac{1}{2}\left(\mathbb{I} + \vec{P}(\omega_q)\cdot\vec{\sigma}\right)$$

Relations between entanglement measures

Entanglement entropy has a one-to-one, inverse relationship with the magnitude of the polarization vector

$$S(P_q) = -\frac{1-P_q}{2}\log\left(\frac{1-P_q}{2}\right) - \frac{1+P_q}{2}\log\left(\frac{1+P_q}{2}\right)$$

with $P_q = |\vec{P}(\omega_q)|$

- $P = 1 \iff S = 0$ (Unentangled)
- $P = 0 \iff S = \log(2)$ (*Maximally* Entangled)

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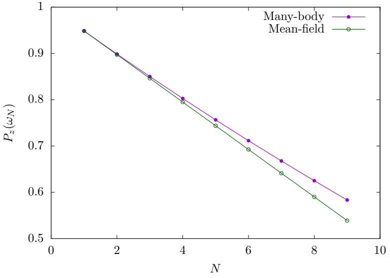
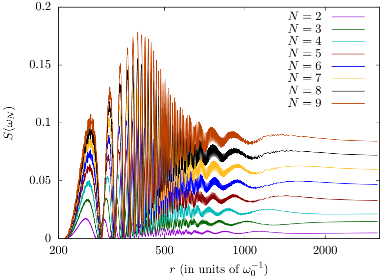
Example: evolution of all-electron flavor initial state

Comparison of many-body and mean-field calculations

- System with frequencies $\omega_1, \ldots, \omega_N$ where $\omega_p = p\omega_0$
- Evolve from $|\Psi_0\rangle = |\nu_e \ldots \nu_e\rangle$ for systems of varying sizes $(N = 2, \ldots, 9)$
- As $\mu \sim 0$ $(r \gg R_\nu)$, H diagonal in mass-basis, therefore plot final spectra in the mass-basis: $P_z = n(\nu_1) - n(\nu_2)$



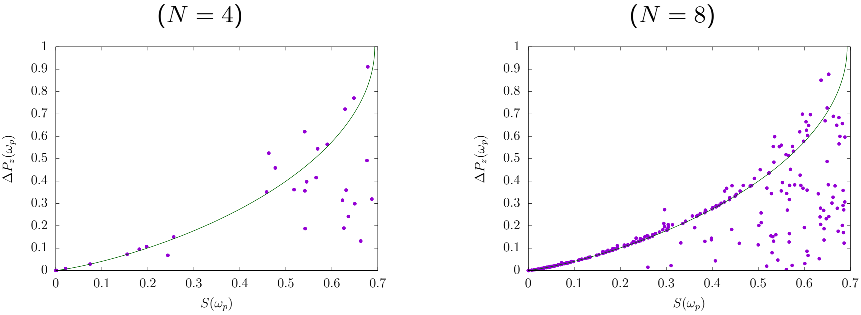


NB: In the mean-field case, $S = 0$ *always*

Correlation of P_z-discrepancies and entanglement entropy

Calculate $\Delta P_z(\omega) \equiv |P_z^{\mathrm{MF}}(\omega) - P_z^{\mathrm{MB}}(\omega)|$ at $r \gg R_\nu$ (i.e., $\mu \approx 0$)

- For $N = 4$: all initial conditions with definite flavor ν_e, ν_x (e.g., $|\nu_e, \nu_x, \nu_x, \nu_x\rangle$)
- For $N = 8$: same ICs as $N = 4$, but with four additional ν_e appended to left or right of spectrum

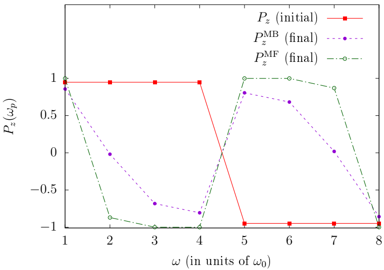


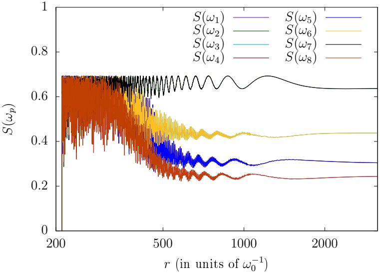
Trendline: $y(S) \equiv P^{\mathrm{MF}}(S) - P^{\mathrm{MB}}(S) = 1 - P(S)$

Example: initial condition with both neutrino flavors

Comparison of final P_z spectra between many-body and mean-field

- Evolve $|\Psi_0\rangle = |\nu_e\nu_e\nu_e\nu_e\nu_x\nu_x\nu_x\nu_x\rangle$ until $r \gg R_\nu$



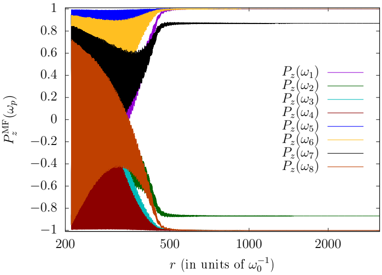


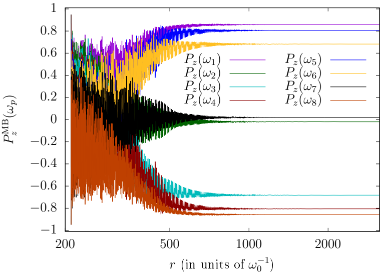
[Cervia et al., Phys. Rev. D 100, 083001 (2019)]

Spectral swap-like features persist in the many-body calculations, but are less sharp relative to mean-field calculations

Comparison of P_z evolution with r

- Same initial conditions, $|\Psi_0\rangle = |\nu_e\nu_e\nu_e\nu_e\nu_x\nu_x\nu_x\nu_x\rangle$



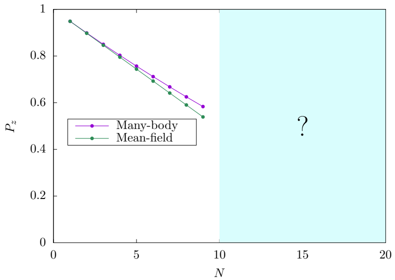


[Cervia et al., Phys. Rev. D 100, 083001 (2019)]

Conclusions

- Calculations of collective neutrino flavor evolution typically rely on a 'mean-field', i.e., effective one-particle description
- Important to test the efficacy and/or limitations of the mean-field by performing many-body calculations
- Evolution in the many-body case can be studied by calculating the eigenvalues and eigenvectors of the Hamiltonian by solving the Bethe Ansatz equations (or an equivalent set of equations)
- For certain simple systems, qualititive differences in flavor evolution observed between many-body and mean-field treatments, resulting from entangled states which are absent in the mean-field limit

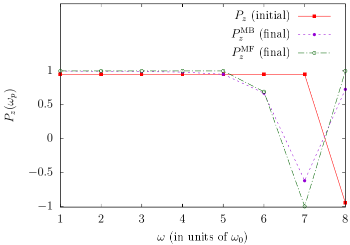
Future Work

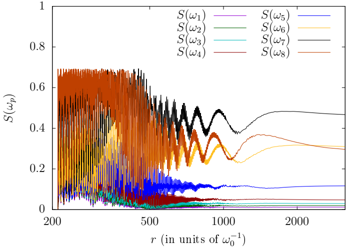


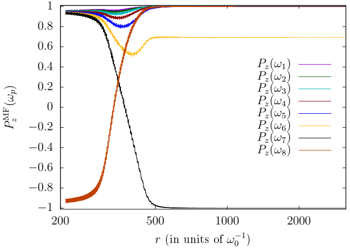
- Next steps in calculations
  - Larger N + inclusion of $\bar{\nu}$ $(\omega < 0)$
  - Matter (MSW) potential
  - Multiple neutrinos in frequency bins $(j_p > 1/2)$
  - Beyond single-angle approximation $\mu \to \mu_{\mathbf{pq}}$
- Hierarchical incorporation of multi-particle correlations
- Couple to baryons—how are nucleosynthetic yields affected?

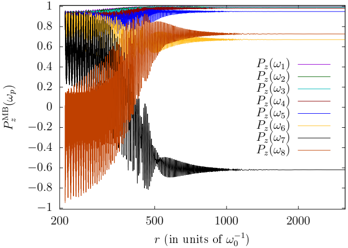
Bonus slides

Different initial condition: $|\Psi_0\rangle = |\nu_e\nu_e\nu_e\nu_e\nu_e\nu_e\nu_e\nu_x\rangle$





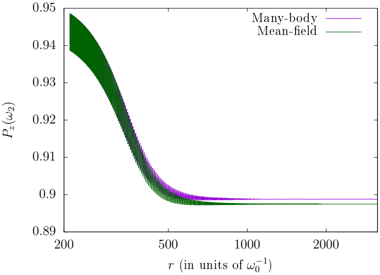


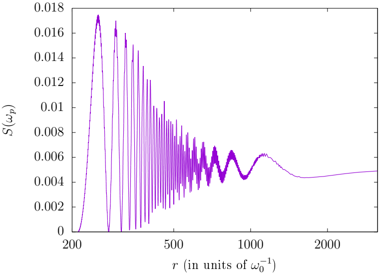


Comparison of Intermediate P_z Spectra

While $r \gtrsim R_\nu$, $N = 2$ mono-flavor initially

- $|\Psi_0\rangle = |\nu_e \nu_e\rangle$, and observe P_z before $r \gg R_\nu$

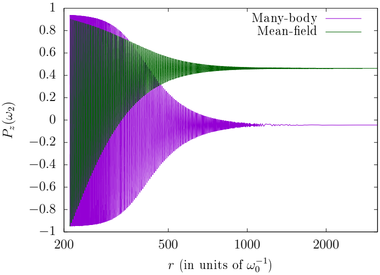


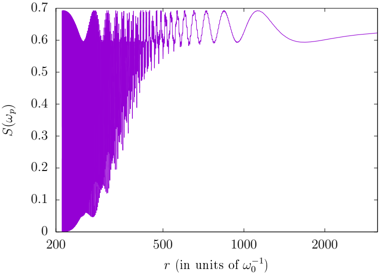


Comparison of Intermediate P_z Spectra

While $r \gtrsim R_\nu$, $N = 2$ different-flavor initially

- $|\Psi_0\rangle = |\nu_e \nu_x\rangle$, and observe P_z before $r \gg R_\nu$





Entanglement in Individual Eigenstates

- Eigenstates for $N = 5$, entanglement of N-th ν with the rest
- Hightest/lowest-weight states are trivial