Advanced Focusing Concepts for Next-Generation Accelerators and Storage Rings

Tim Zolkin[1]

[1]Fermi National Accelerator Laboratory

9 December, 2019

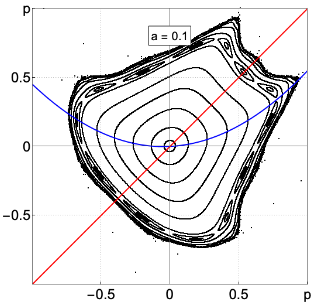
Goals in focusing concepts

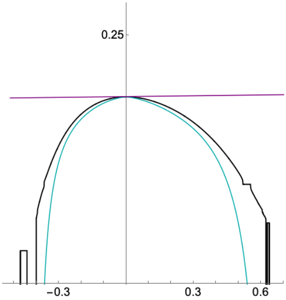
Light sources

Produce globally linear map (removal of all nonlinear resonances, frequency are independent of the amplitude)

Proton machines (machines with instability)

Nonlinear optics (frequency-dependent amplitude) and integrals of motion





What do we know now?

Implemented Advanced Focusing Concepts

- Round beams at VEPP-2000 (BINP)
  [S. Danilov and E. A. Perevedentse])
- IOTA (FNAL)
  [S. Danilov and S. Nagaitsev])

Advanced Focusing Concepts in development

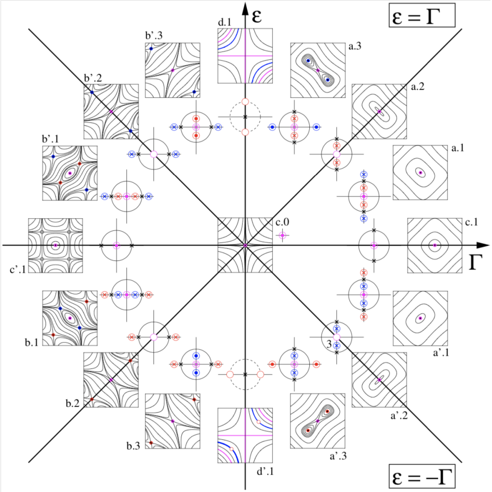
- McMillan lens
  - 1D McMillan Lens [E. McMillan]
  - Near-integrable electrostatic McMillan lens [S. Danilov]
  - Axially symmetric electron lens [S. Danilov]
- Lattice design based on the Yoshida integrator [S. S. Baturin]

1D McMillan Lens

$$q' = p$$

$$p' = -q + \frac{2\,\epsilon}{p^2 + \Gamma}$$

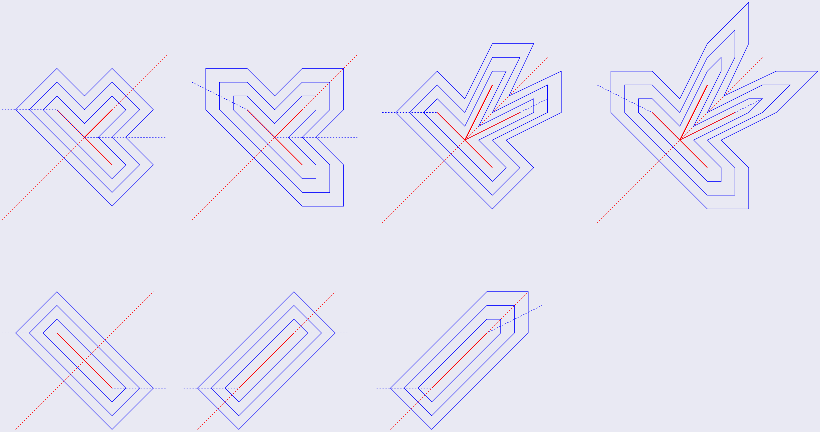
- 1D map
- Hard to generalize to higher dimensions
- Not very stable to perturbations



Bifurcation diagram of octupole McMillan map.

1D McMillan Lens

New 1D mappings



Mappings with polygon invariants [T. Zolkin, S. Nagaitsev]

1D McMillan Lens

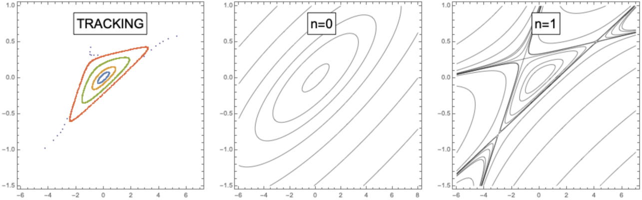
Perturbation theory and McMillan Map

For the general map

$$q' = (a\,p^2 + b\,p\,q + c\,q^2) + (d\,p^3 + e\,p^2 q + f\,p\,q^2 + g\,q^3)\varepsilon + \ldots$$
$$p' = (\bar{a}\,p^2 + \bar{b}\,p\,q + \bar{c}\,q^2) + (\bar{d}\,p^3 + \bar{e}\,p^2 q + \bar{f}\,p\,q^2 + \bar{g}\,q^3)\varepsilon + \ldots$$

one can show that there is an approximate integral of motion

$$\mathrm{K} = (\alpha\,p^2 + \beta\,p\,q + \gamma\,q^2) + (\delta\,p^2 q + \epsilon\,p\,q^2)\,\varepsilon + \zeta\,p^2\,q^2\,\varepsilon^2$$

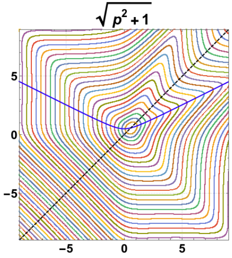


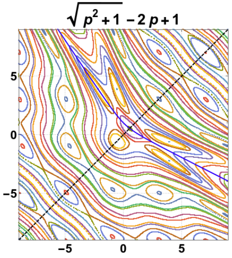
Mu2e delivery ring: tracking vs perturbation theory.

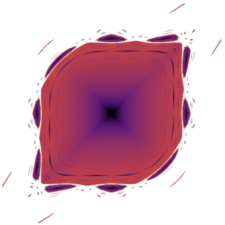
1D McMillan Lens

Application of mappings with polygon invariants

- Construction of near-integrable Cohen-like mappings
- Foundation for another perturbation theory
- Understanding of topology for near-resonance mappings with smooth force function

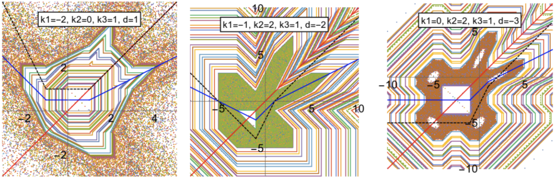






Questions we should Ask Ourselves

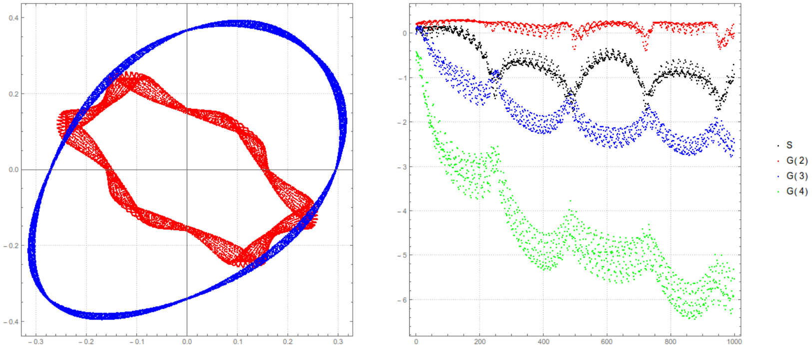
- Do we really need an integrable system or near-integrable system is enough?
- How we will define near-integrable system?
- Can we have a nonintegrable system with isolating integral in 6D? Is it possible to create chaotic system like that?
- Can we create a structurally stable system?
- How we define perturbations for structural stability? Can we create a strange attractor?



Near-integrable electrostatic McMillan lens

State of art simulation by I. Morozov

Near-integrable electrostatic McMillan lens



State of art simulation by I. Morozov (BINP)

Axially symmetric electron lens [Slava Danilov]

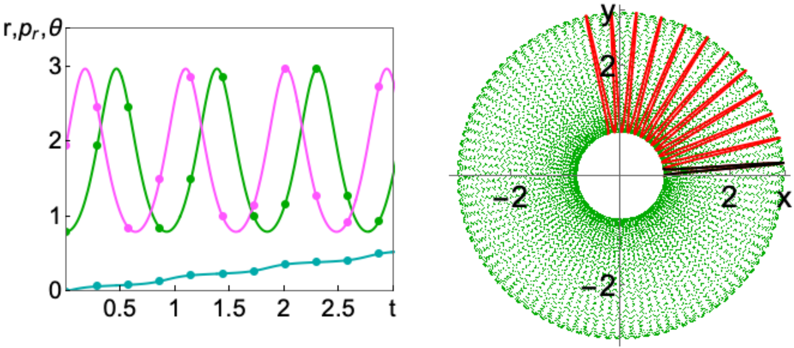
This 4D map can be realized in accelerators by employing the electron lens:

$$\begin{bmatrix} x' \\ p_x' \\ y' \\ p_y' \end{bmatrix} = \begin{bmatrix} \alpha_x x + \beta\, p_x \\ -\gamma_x x - \alpha_x\, p_x + \frac{a\, x'}{b\, r'^2+1} \\ \alpha_y y + \beta\, p_y \\ -\gamma_y y - \alpha_y\, p_y + \frac{a\, y'}{b\, r'^2+1} \end{bmatrix} \to \begin{bmatrix} r' \\ p_r' \\ \theta' \\ p_\theta' \end{bmatrix} = \begin{bmatrix} \sqrt{p_r^2 + \frac{p_\theta^2}{r^2}} \\ -p_r\, \frac{r}{r'} + \frac{a\, r'}{b\, r'^2+1} \\ \theta + \arctan \frac{p_\theta}{r\, p_r} \\ p_\theta \end{bmatrix}$$

It has two integrals of motion

$$\mathcal{K}[r, p_r, p_\theta] = b\, r^2 p_r^2 + r^2 - a\, r\, p_r + p_r^2 + \frac{p_\theta^2}{r^2} \quad \text{and} \quad p_\theta = x\, p_y - y\, p_x.$$

Axially symmetric electron lens



[S. Nagaitsev NAPAC-19 (FERMILAB-POSTER-19-122-DI-SCD)]

Lattice design based on the Yoshida integrator

S.S. Baturin ArXiv:1908.03520

Hamiltonian preserving nonlinear optics

Algorithm

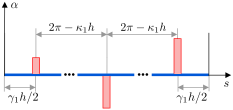
1 . Pick an integrator for a smoth Hamiltonian in normalized coordinates

2. Set optics with the step in phase prescribed by the integrator

3. Set nonlinear magnet strength according to similarity transformation

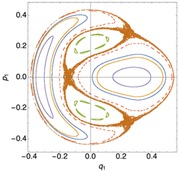
$$\mathbf{B}(s_1) \circ K_h \circ \mathbf{B}^{-1}(s_1)$$

Yoshida lattice with sextupoles - a linear ring with three sextupole magnets that mimics dynamics of a well known Henon-Heiles system based on a 4th order Yoshida integrator.

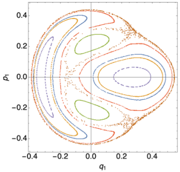


$$\Phi_h^Y = \Phi_{\gamma_3 h} \circ \Phi_{\gamma_2 h} \circ \Phi_{\gamma_1 h},$$

$$\gamma_1 = \gamma_3 = \frac{1}{2 - 2^{1/3}}, \quad \gamma_2 = -\frac{2^{1/3}}{2 - 2^{1/3}}$$



Poincare surface of section q2=0 for the Smooth Hamiltonian



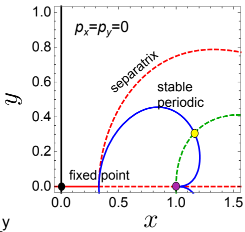
Poincare surface of section q2=0 for the Yoshida lattice

Applying Mathematical Methods in Integrable Hamiltonian Flows to the Analysis and Design of Nonlinear Accelerator Lattices (1)

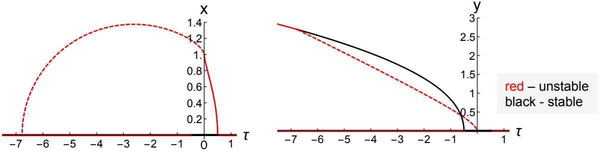
Finding Phase Space Critical Points and Bifurcation Sets

- **Need:** To understand the global, qualitative single-particle dynamics accessible in accelerator lattices based on nonlinear integrable optics.
- **Problem:** Standard approaches to nonlinear dynamics in the accelerator community are perturbative about the the origin, neglect fully 4D or 6D coupling, or require a clever choice of coordinates.
- **Solution:** Differential-geometric methods from the theory of integrable Hamiltonian systems may be applied to locate critical structures (fixed points, periodic orbits, separatrix-like structures in 4D or 6D) using knowledge only of the invariants of motion.

Critical initial conditions at nominal IOTA insert strength



Bifurcations of dynamical fixed points in IOTA vs. magnetic insert strength

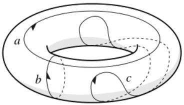


Applying Mathematical Methods in Integrable Hamiltonian Flows to the Analysis and Design of Nonlinear Accelerator Lattices (2)

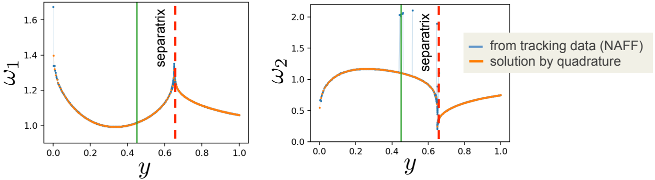
Extracting Characteristic Tunes from Invariants of Motion

- **Need:** To understand analytically the orbital frequency content (nonlinear tune spread), critical for prediction and control of Landau damping.
- **Problem:** Traditional methods for analysis of integrable Hamiltonian systems rely on action-angle coordinates, which are difficult to obtain in explicit form in most cases, and which break down near critical phase space structures.
- **Solution:** Characteristic frequencies may be extracted using sets of path integrals taken over the invariant level sets, requiring only knowledge of the invariants of motion.

Examples of cycles on an invariant torus for computing frequencies

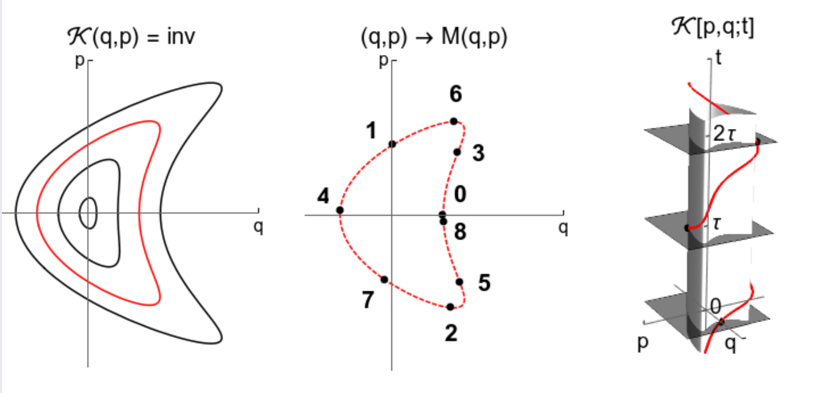


Characteristic frequencies in IOTA along a line of initial conditions in the (x,y) plane



Danilov Theorem

S. Nagaitsev arXiv:1910.08630



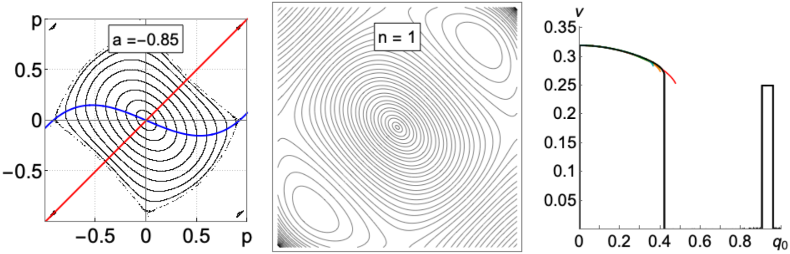
Nonlinear optical functions

$$\mathbf{inv}(s) = \underbrace{\alpha(s)\,p^2 + \beta(s)\,p\,q + \gamma(s)\,q^2}_{\text{Courant}-\text{Snyder}} + \underbrace{\delta(s)\,p^2\,q + \epsilon(s)\,p\,q^2}_{\textit{sextupoles}} +$$

$$+ \underbrace{\zeta(s)\,p^2\,q^2}_{\textit{octupoles}} + \underbrace{\eta(s)\,\mathrm{c.s.}^2}_{\textit{2nd order correction}}$$

- Sextupole and octupole terms are in the form of McMillan integrable mappings
- Estimate of dynamical aperture near 1st, 2nd, 3rd and 4th order resonances (critical points of the invariant)
- Distortion of the ellipse trajectories on larger amplitudes ($\triangle$, $\square$, C- or S-shapes)
- Amplitude dependent betatron frequency $\mu(q_0, p_0)$

Example for Hénon octupole map



Summary

Road map for integrable optics

- Find systems integrable in continuous space of parameters (ideally all 3 DOF)
- Find systems with large spread of frequencies
- Inclusion of SC

Road map for analytical research

- Development of general analytical tools: perturbation theories for near integrable and extraction of dynamics from invariants of motion for integrable systems.
- Better understanding of nonlinear coupling
- Use of advanced tools (SALI, GALI, genetic algorithms)

Summary

Road map for theoretical understanding

- Integrability vs Near-Integrability vs Isolating integrals
- Define perturbations for structural stability studies
- Integrable systems vs Ergodic systems vs Strange Attractors

Many thanks to S. S. Baturin, C. Mitchell, I. Morozov, S. Nagaitsev and J. Eldred.