An analytical approach to neutrino propagation through the Earth

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Introduction

- Neutrino oscillations experiments can be interpreted in a 3ν framework.
- Very good knowledgement about the values of the oscillation parameters. $\theta_{13} \neq 0$ has now open the possibility to measure the CP-violating phase.
- An accurate description of matter neutrino oscillations is an important ingredient in the analysis of the data.
- We aim for an analytic solution:
  - Help to save the CPU time;
  - Give significant insight into the physics of the problem;
  - Better understanding of the dependence of the oscillations on the neutrino parameters and the properties of the medium.

Neutrino State and its evolution

We write the neutrino state in time as

$$|\psi(t)\rangle = \sum_{\alpha} \psi_\alpha(t)|\nu_\alpha\rangle, \qquad \alpha = e, \mu, \tau,$$

which evolves through the evolution operator $\mathcal{U}(t, t_0)$

$$\begin{pmatrix} \psi_e(t) \\ \psi_\mu(t) \\ \psi_\tau(t) \end{pmatrix} = \mathcal{U}(t, t_0) \begin{pmatrix} \psi_e(t_0) \\ \psi_\mu(t_0) \\ \psi_\tau(t_0) \end{pmatrix},$$

by the equation

$$i\hbar \frac{\partial}{\partial t}\mathcal{U}(t, t_0) = H(t)\mathcal{U}(t, t_0), \quad \mathcal{U}(t_0, t_0) = I,$$

Neutrino State and its evolution

The Hamiltonian is

$$H(t) = U \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta_{21} & 0 \\ 0 & 0 & \Delta_{21} \end{pmatrix} U^\dagger + \begin{pmatrix} V(t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Delta_{ij} = \frac{\Delta m^2_{ij}}{2E}, \quad V(t) = V_e(t) - V_\mu(t) = \sqrt{2} G_F N_e(t), \quad V_\mu(t) = V_\tau(t)$$

and the mixing matrix in vacuum is the product of rotation matrices

$$U = \mathcal{O}_{23} \Gamma \mathcal{O}_{13} \mathcal{O}_{12} \Gamma^*.$$

The Hamiltonian is diagonalized through

$$H_D(t) = U_m(t) H(t) U_m^\dagger, \quad U_m(t) \approx \mathcal{O}_{23} \Gamma \mathcal{O}^m_{13} \mathcal{O}^m_{12} \Gamma^*$$

Neutrino State and its evolution

$$E_1 \approx \tfrac{1}{2}\left[\left(\Delta_{21} + V c_{13}^2\right) - \Delta_{21}^m(t)\right],$$
$$E_2 \approx \tfrac{1}{2}\left[\left(\Delta_{31} + \Delta_{21} c_{12}^2 + V s_{13}^2\right) + \left(\Delta_{21}^m(t) - \Delta_{32}^m(t)\right)\right],$$
$$E_3 \approx \tfrac{1}{2}\left[\left(\Delta_{31} + \Delta_{21} s_{12}^2 + V\right) + \Delta_{32}^m(t)\right],$$

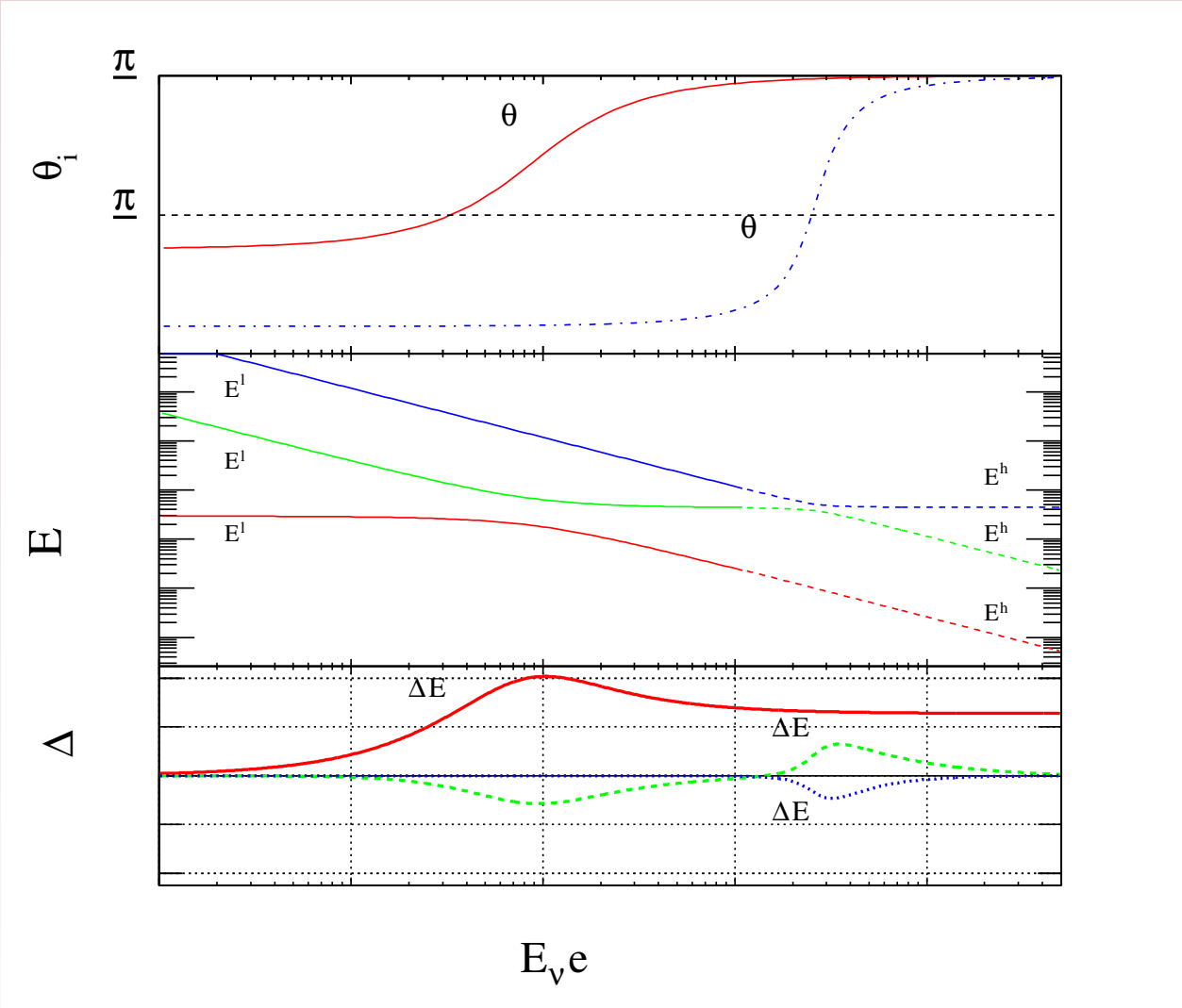
with $c_{ij} \equiv \cos\theta_{ij}$, $s_{ij} \equiv \sin\theta_{ij}$,

$$\Delta_{21}^m = c_{13}^2 \sqrt{(V - V_l^R)^2 + (V_l^R \tan 2\theta_{12})^2},$$
$$\Delta_{32}^m = \sqrt{(V - V_h^R)^2 + (V_h^R \tan 2\theta_{13})^2},$$

$$V_l^R = \Delta_{21}(\cos 2\theta_{12}/c_{13}^2) \qquad V_h^R = (\Delta_{31} - \Delta_{21} s_{12}^2)\cos 2\theta_{13},$$

$$\tan 2\theta_{12,13}^m = \frac{V_{l,h}^R \tan 2\theta_{12,13}}{V_{l,h}^R - V}.$$

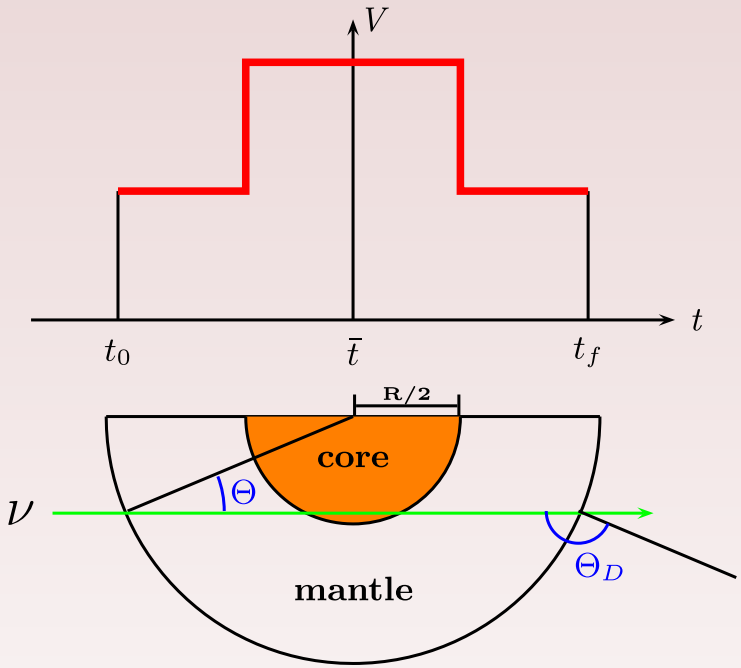
Neutrino State and its evolution



$V = 4{,}54 \times 10^{-4}$ eV^2/GeV (Earth's core), $\Delta m^2_{21} = 7{,}59 \times 10^{-5}$ eV^2, $|\Delta m^2_{31}| = 2{,}35 \times 10^{-3}$ eV^2, $\theta_{12} = 34°$, $\theta_{13} = 8{,}9°$, and $\delta = 0$.

Magnus Approximation

For a symmetric potential modelling the Earth



we find the solution for the evolution of the neutrino system by calculating

$$\mathcal{U}_P(t_f, t_0) = \mathcal{U}_l(t_f, \bar{t})\, \mathcal{U}_h(t_f, t_0)\, \mathcal{U}_l(\bar{t}, t_0),$$

Magnus Approximation

writing $\mathcal{U}_{l,h}$ of the form $\mathcal{U}(t,t_0) = \exp[\Omega(t,t_0)]$, with Ω given by $\sum_{n=1}^{\infty} \Omega_n \rightarrow$ Magnus expansion. We find

$$\mathcal{U}_P = \begin{pmatrix} u_{11} & u_{12} & 0 \\ -u_{12}^* & u_{11}^* & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & v_{11} & v_{12} \\ 0 & -v_{12}^* & v_{11}^* \end{pmatrix} \times \begin{pmatrix} u_{11} & u_{12} & 0 \\ -u_{12}^* & u_{11}^* & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$u_{11} = \cos\xi^l - i\sin\xi^l \frac{\xi^l_{(2)}}{\xi^l}, \qquad v_{11} = \cos\xi^h - i\sin\xi^h \frac{\xi^h_{(2)}}{\xi^h}$$

$$u_{12} = i\sin\xi^l \frac{\xi^l_{(1)}}{\xi^l}, \qquad v_{12} = i\sin\xi^h \frac{\xi^h_{(1)}}{\xi^h} \exp[-i\phi_{32}(\bar{t})]. \tag{1}$$

Magnus Approximation

$$\xi^l_{(1)} = i \int_{t_0}^{\bar{t}} dt\, \dot{\theta}^m_{12}(t) \exp(-i\phi_{21}^{t_0 \to t})$$

$$\xi^l_{(2)} = \int_{t_0}^{\bar{t}} dt \int_{t_0}^{t} dt'\, \dot{\theta}^m_{12}(t)\, \dot{\theta}^m_{12}(t') \sin \phi_{21}^{t \to t'},$$

$$\xi^h_{(1)} = 2 \int_{\bar{t}}^{t_f} dt' \dot{\theta}^m_{13}(t') \sin \phi_{32}^{\bar{t} \to t'},$$

$$\xi^h_{(2)} = \int_{t_0}^{t_f} dt \int_{t_0}^{t} dt'\, \dot{\theta}^m_{13}(t)\, \dot{\theta}^m_{13}(t') \sin \phi_{31}^{t \to t'},$$

$$\xi^{l,h} = \sqrt{(\xi^{l,h}_{(1)})^2 + (\xi^{l,h}_{(2)})^2}.$$

Oscillation Probability

The oscillation probability $P(\nu_\mu \to \nu_e) = |\psi_{\mu\to e}|^2$, with the amplitude

$$\psi_{\mu\to e} = e^{-i\alpha_1}\, U^m_{e1}(t_f)\left[U^{m*}_{\mu 1}(t_0)a_{11} + U^{m*}_{\mu 2}(t_0)a_{12} + U^{m*}_{\mu 3}(t_0)a_{13}e^{-i\delta}\right]$$

$$+\, e^{-i\alpha_2}\, U^m_{e2}(t_f)\left[U^{m*}_{\mu 1}(t_0)a_{21} + U^{m*}_{\mu 2}(t_0)a_{22} + U^{m*}_{\mu 3}(t_0)a^*_{23}e^{-i\delta}\right]$$

$$+\, e^{-i(\alpha_3-\delta)}\, U^m_{e3}(t_f)\left[U^{m*}_{\mu 1}(t_0)a^*_{31} - U^{m*}_{\mu 2}(t_0)a^*_{32} + U^{m*}_{\mu 3}(t_0)a^*_{33}e^{-i\delta}\right]$$

with

Oscillation Probability

$$a_{11} = (u_{11})^2 + (u_{12}^*)^2 v_{11} e^{-i\phi_{21}^{t_0 \to t_f}},$$

$$a_{22} = (u_{11}^*)^2 v_{11} + (u_{12})^2 e^{i\phi_{21}^{t_0 \to t_f}}, \quad a_{33} = v_{11}^*,$$

$$a_{12} = a_{21} e^{-i\phi_{21}^{t_0 \to t_f}} = u_{11} u_{12} + u_{11}^* u_{12}^* v_{11} e^{-i\phi_{21}^{t_0 \to t_f}},$$

$$a_{13} = -u_{12}^* v_{12} e^{-i\phi_{21}^{t_0 \to t_f}}, \qquad a_{31} = u_{12}^* v_{12}^*,$$

$$a_{23} = u_{11}^* v_{12}, \qquad a_{32} = -u_{11}^* v_{12}^*.$$

Oscillation Probability: Results

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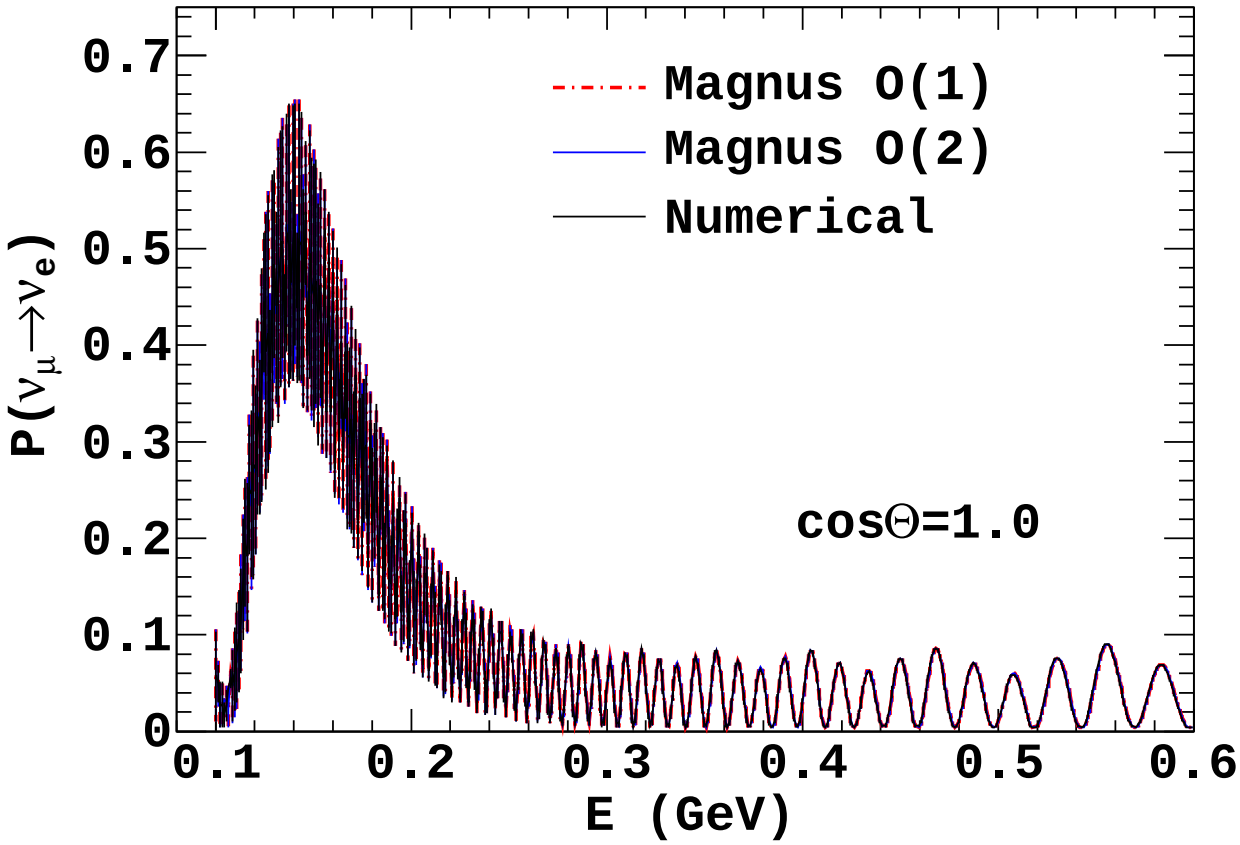
Low Energy neutrinos:



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$\theta_{12} = 34{,}0°$,
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Oscillation Probability: Results

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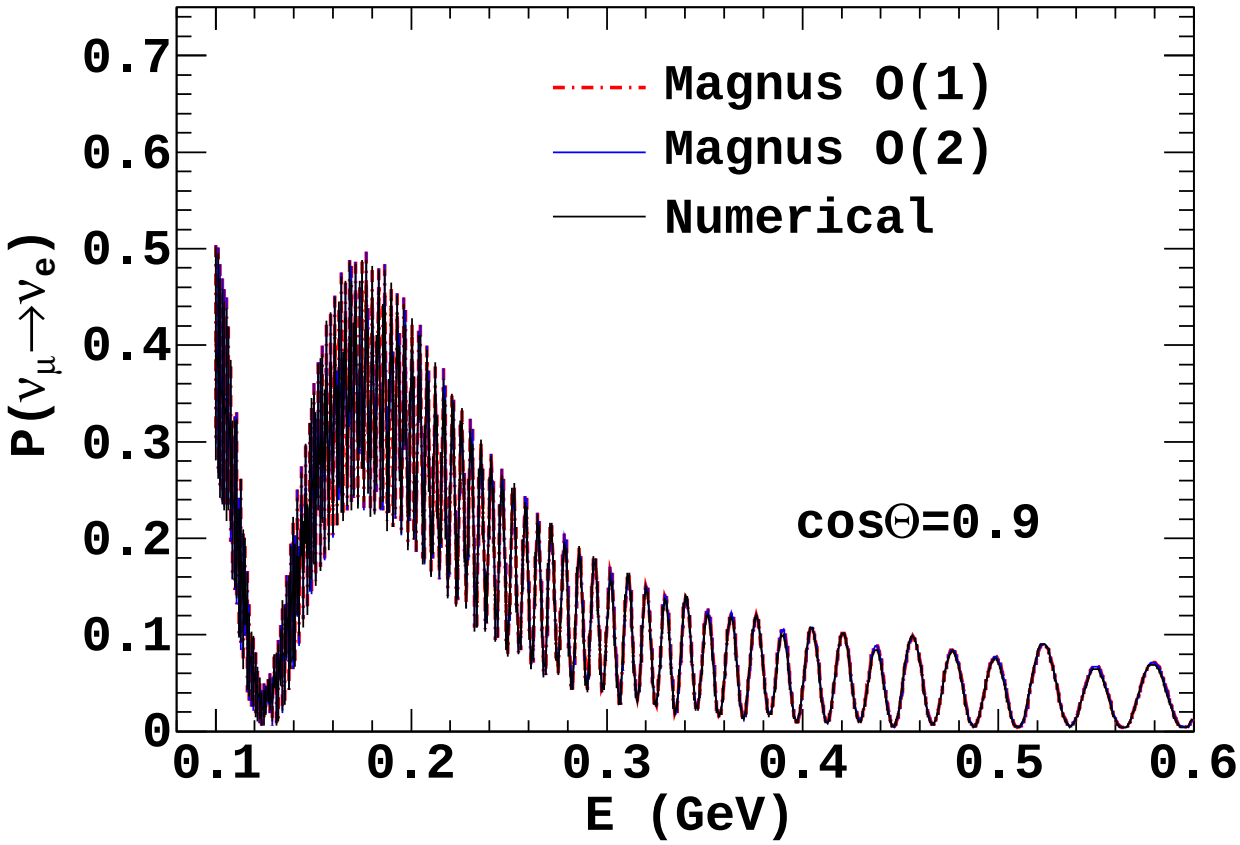

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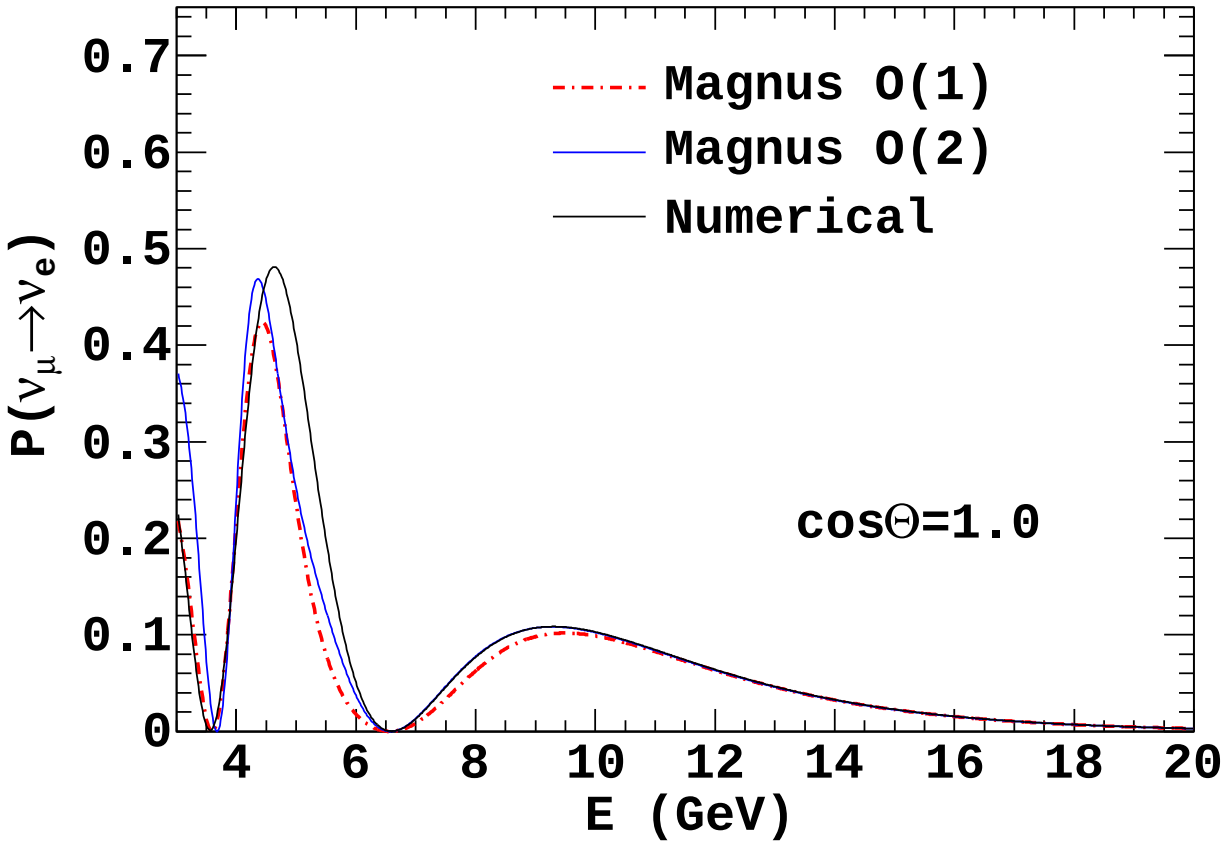

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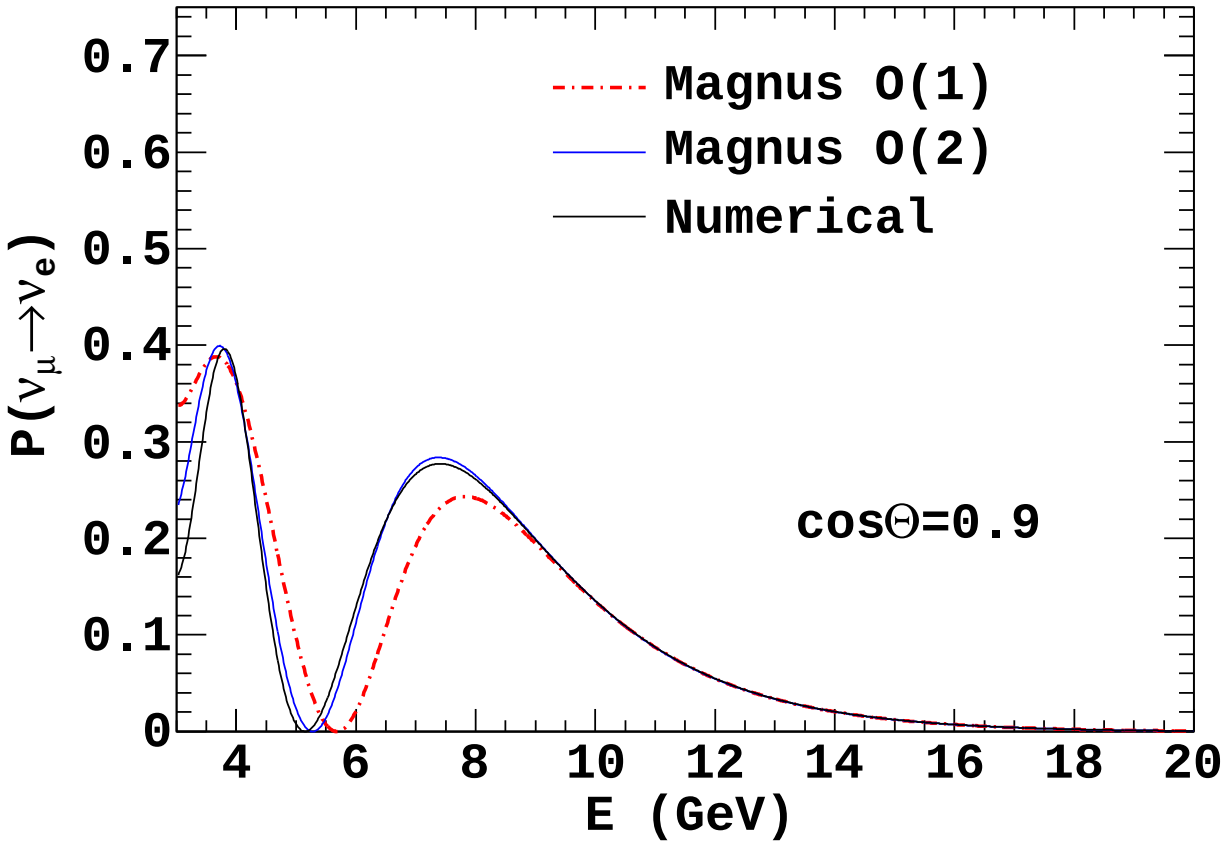

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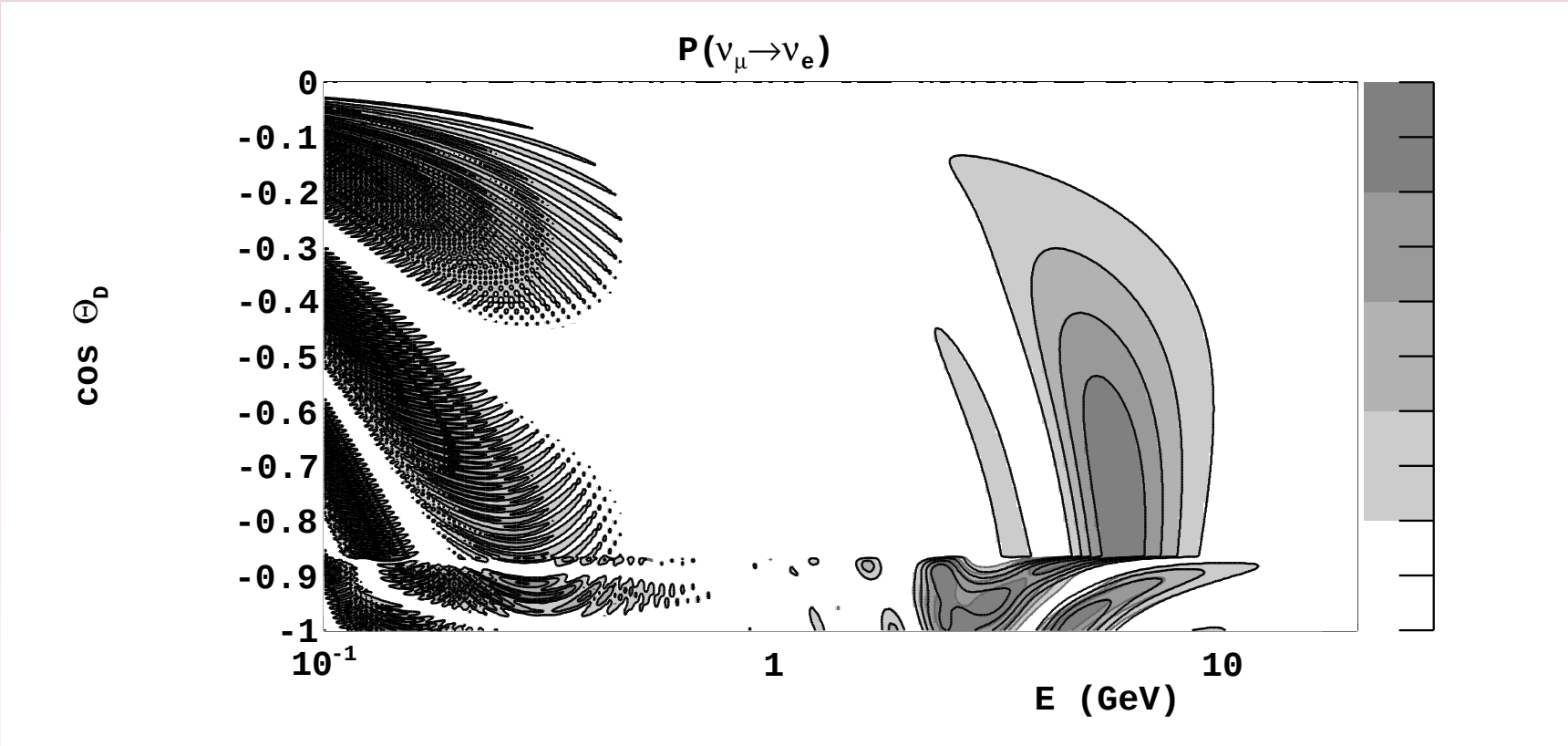

$\cos\Theta_D = -1$ and $\cos\Theta_D = 0$ correspond to vertically upward (traversing the Earth) and horizontal (tangential to Earth's surface) directions, respectively.

Conclusions

- The Magnus expansion for the evolution operator implemented in the adiabatic basis gives an elegant and efficient formalism to describe three neutrino oscillations in a medium with varying density.
- The product of the solutions for the low and high energy regimes found with this method, renders simple semi-analytical formulas for the transition probabilities.
- When these formulas are applied to neutrinos traversing the Earth, they agree well with numerical calculations.