Lattice QCD at non-zero baryon density

How far can we go ?

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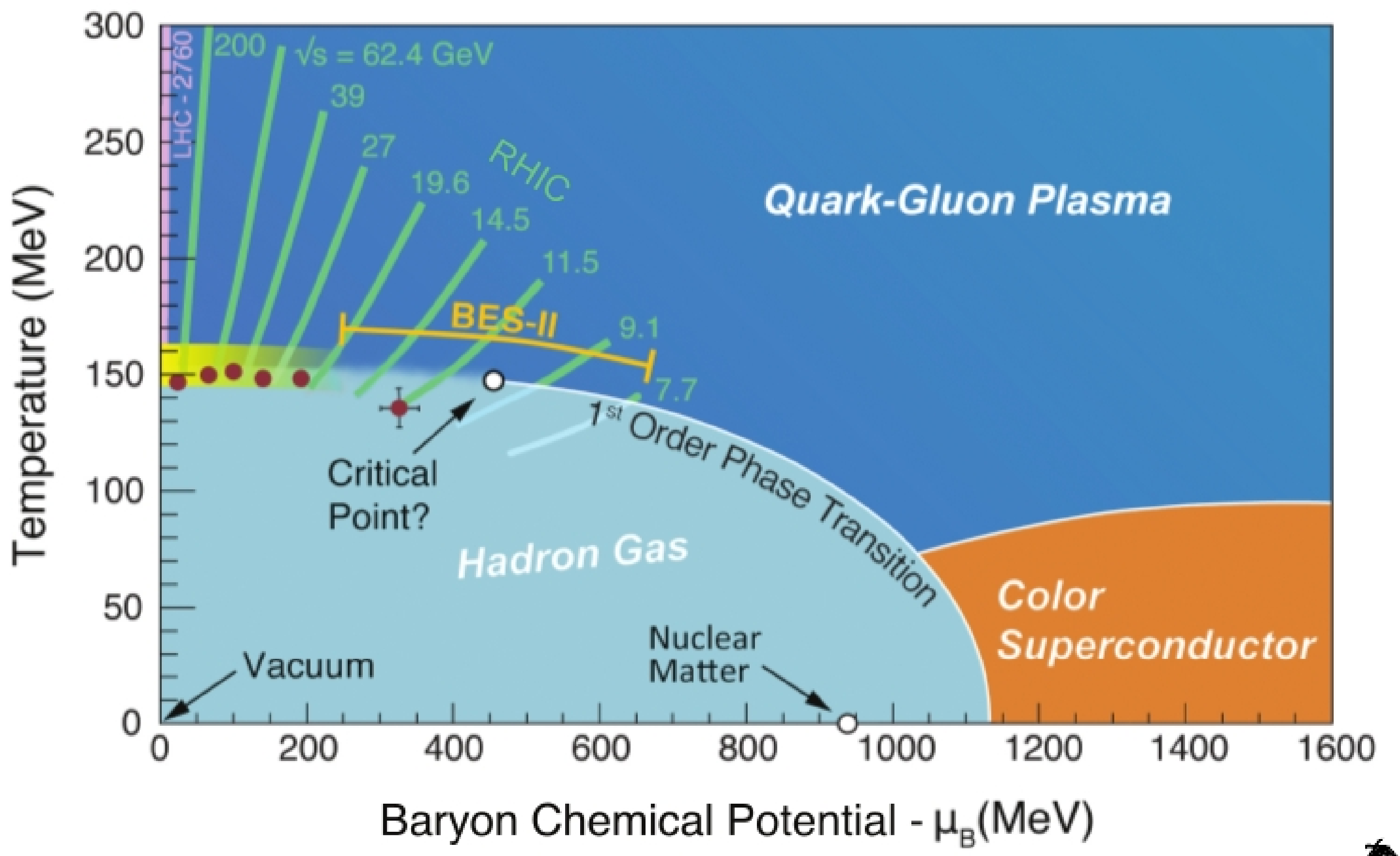
Lattice QCD goals

- determine phase boundaries; characterize bulk thermodynamics and fluctuations of conserved charges on the freeze-out line
- provide the equation of state for BES
- (constrain) the location of the critical point

CRC-TR 211 Strong-interaction matter under extreme conditions
DFG Deutsche Forschungsgemeinschaft

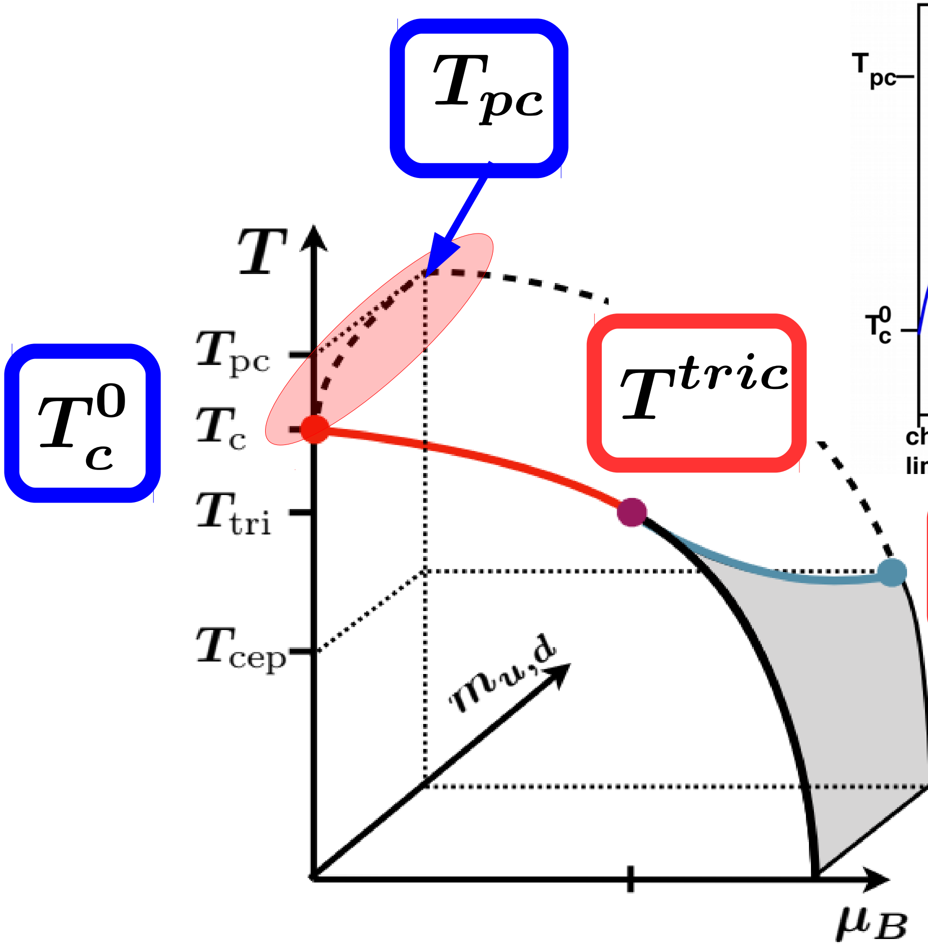
Exploring the phase diagram of strong-interaction matter

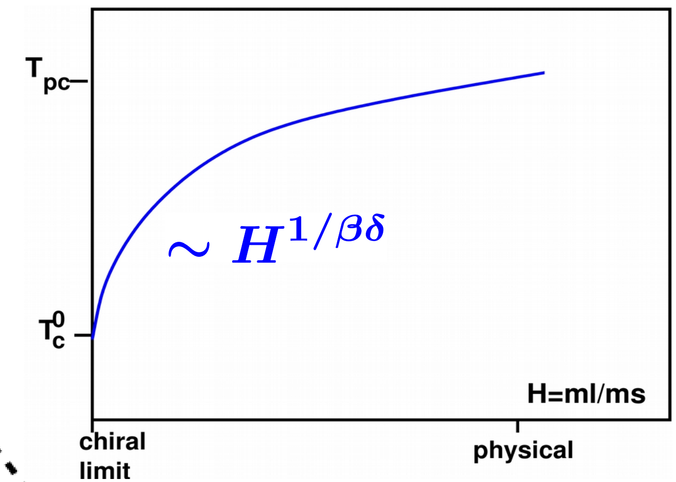


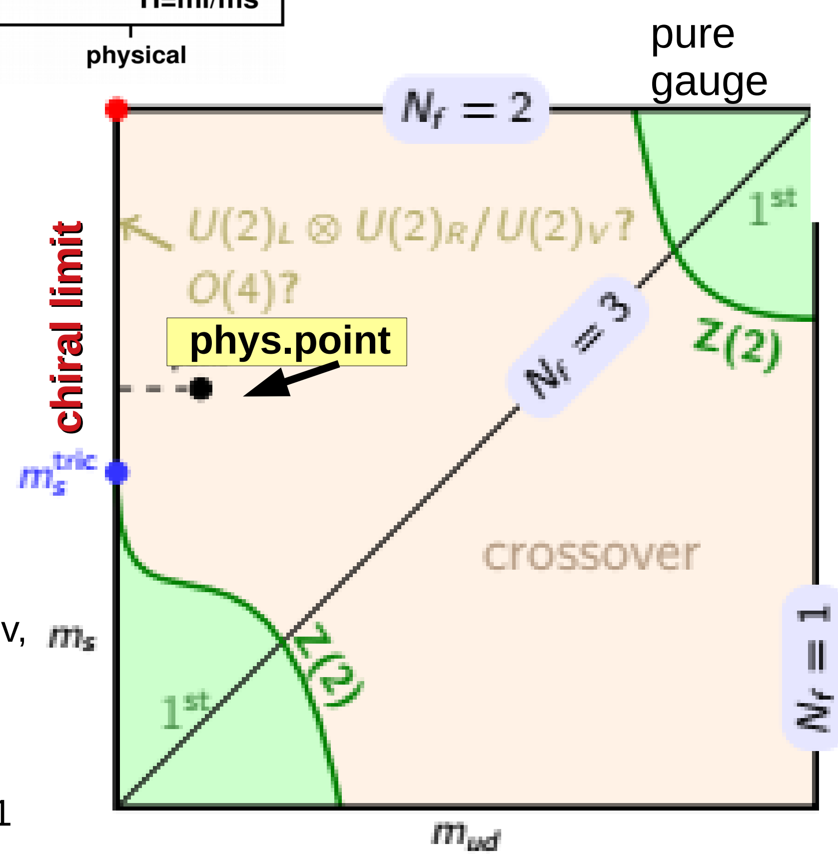
Where is the
critical point?

Phases of strong-interaction matter

determination of T_c^0 puts an upper limit on T^{CEP}







Random Matrix Model — A. Halasz, A.D. Jackson, R.E. Shrock, M.A. Stephanov, J.J.,M. Verbaarschot, Phys. Rev. D58 (1998) 096007

QCD — M. Stephanov, Phys. Rev. D73 (2006) 094508

NJL — M. Buballa, S. Carignano, Phys. Lett. B791 (2019) 361

Critical behavior in QCD

- close to the chiral limit thermodynamics in the vicinity of the QCD transition(s) is controlled by a universal scaling function

singular — regular

$$\frac{p}{T^4} = \frac{1}{VT^3} \ln Z(V,T,\vec{\mu}) = -h^{(2-\alpha)/\beta\delta} f_f(t/h^{1/\beta\delta}) - f_r(V,T,\vec{\mu})$$

$$t \sim \frac{T-T_c}{T_c} + \kappa_2 \left(\frac{\mu_q}{T}\right)^2 \qquad h \sim \frac{m_q}{T_c}$$

Pseudo-critical temperatures

response functions
2nd order cumulants

- magnetic | mixed | thermal

$$\frac{\partial^2 \ln Z}{\partial h^2} \qquad \frac{\partial^2 \ln Z}{\partial h\, \partial t} \qquad \frac{\partial^2 \ln Z}{\partial t^2}$$

$$\sim \left(\frac{m_l}{T_c}\right)^{1/\delta-1} \qquad \sim \left(\frac{m_l}{T_c}\right)^{(\beta-1)/\beta\delta} \qquad \sim \left(\frac{m_l}{T_c}\right)^{-\alpha/\beta\delta}$$

$$\sim -0.79 \qquad \sim -0.34 \qquad \sim +0.11$$

divergence: strong | moderate | none

O(4) critical exponents

$\alpha = -0.21$

$\beta = 0.38$

$\delta = 4.82$

Chiral observables in QCD

– chiral condensate: $\langle\bar\psi\psi\rangle_q = \frac{\partial P/T}{\partial m_q/T}$, $\langle\bar\psi\psi\rangle_l = (\langle\bar\psi\psi\rangle_u + \langle\bar\psi\psi\rangle_d)/2$

– chiral order parameter: $M = \frac{2}{f_K^4}\left[m_s\langle\bar\psi\psi\rangle_l - m_l\langle\bar\psi\psi\rangle_s\right]$

$$m_l = (m_u + m_d)/2$$

– chiral susceptibility: $\chi_M = m_s\left(\frac{\partial M}{\partial m_u} + \frac{\partial M}{\partial m_d}\right)$ **magnetic**

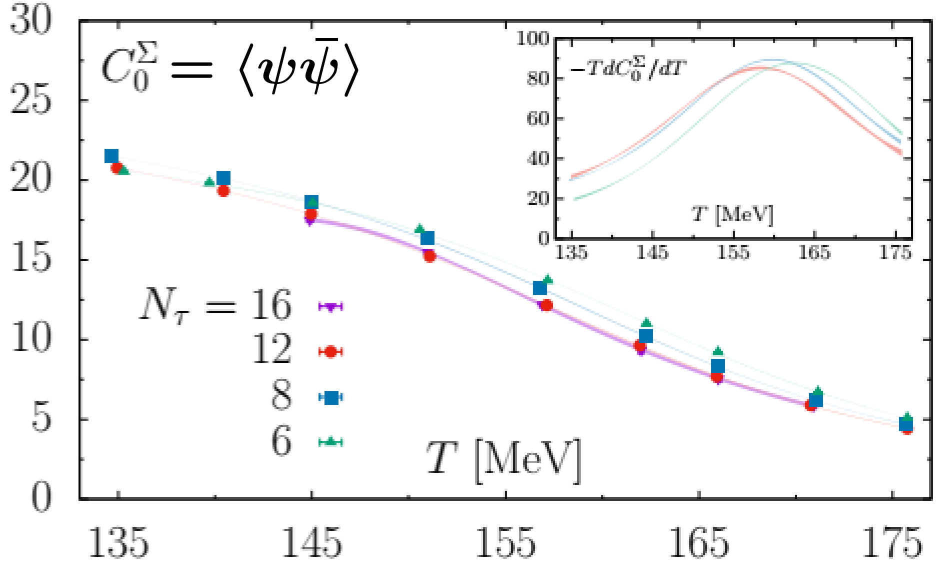
– mixed chiral susceptibility: $\chi_t = T\frac{\partial M}{\partial T}$ **mixed**

– conserved charge fluctuations: $\chi_X = T^2 \frac{\partial^2 P/T^4}{\partial \mu_X^2}\Big|_{\mu_X=0}$ **thermal**

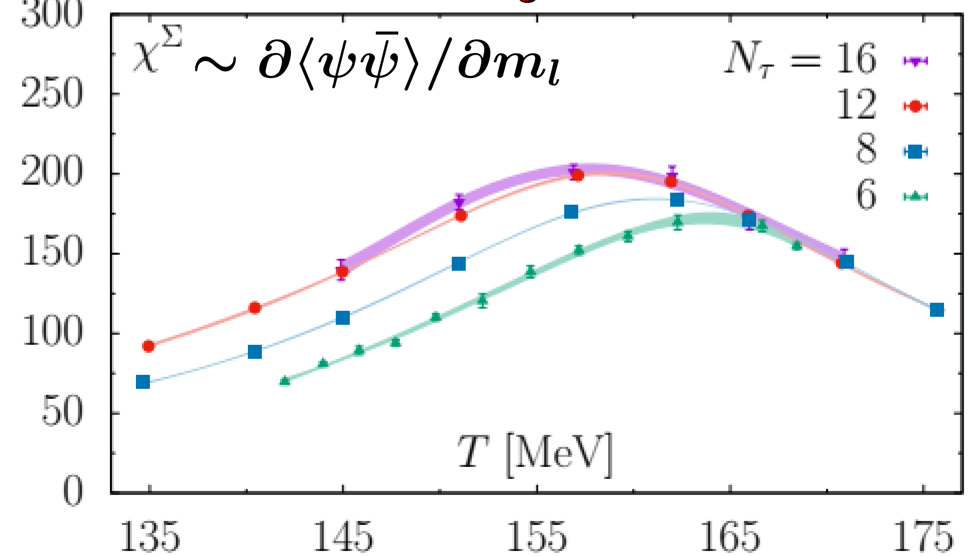
$$X = B,\ S, ...$$

Pseudo-critical temperatures from chiral observables

mixed

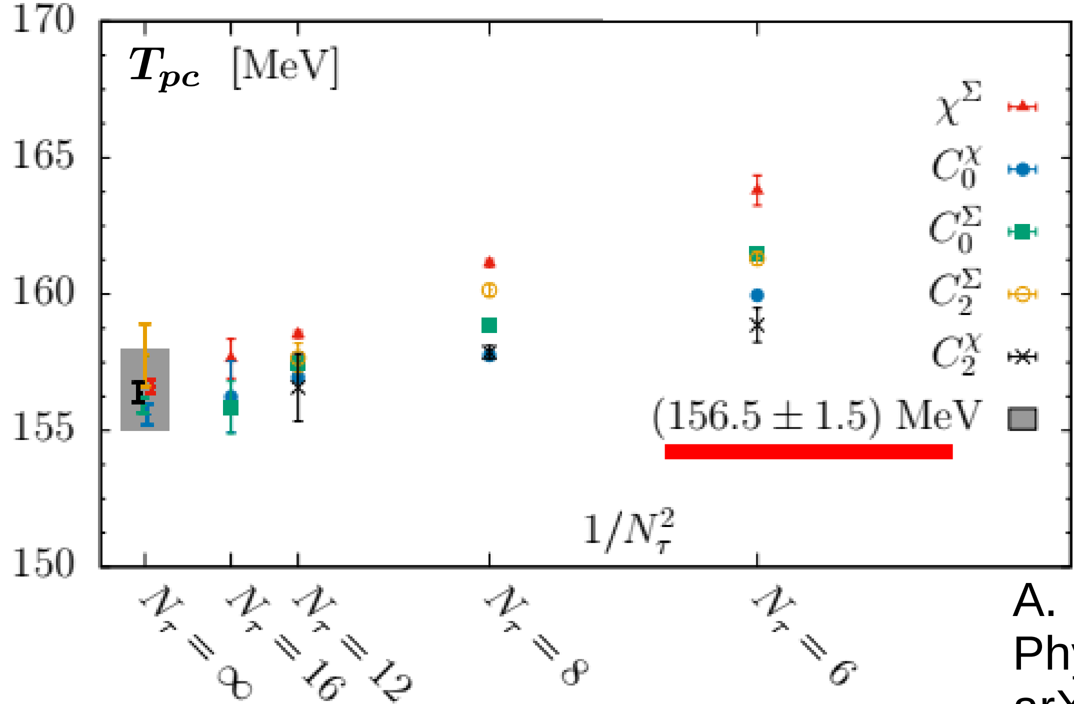


magnetic



physical
light & strange
quark masses;

continuum
extrapolated



$\chi^\Sigma \Leftrightarrow \chi_M$

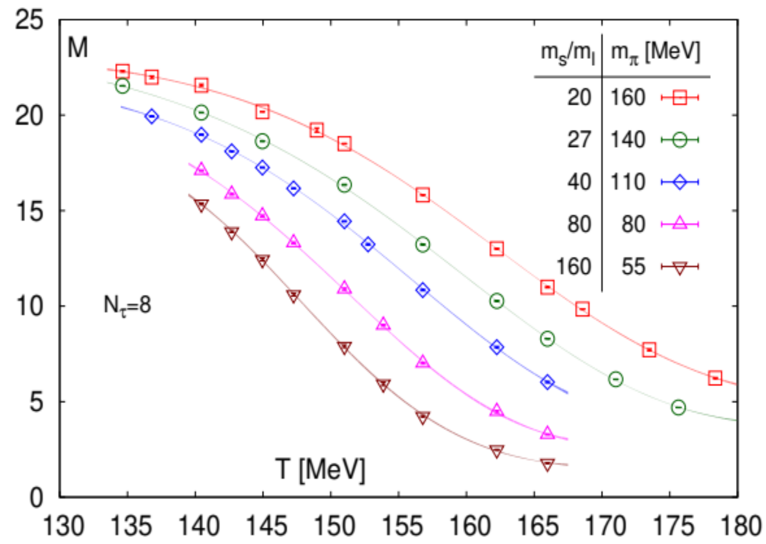
$C_0^\Sigma \Leftrightarrow T(\partial M/\partial T)$

$C_2^\Sigma \Leftrightarrow \partial^2 M/\partial(\mu_B/T)^2$

A. Bazavov et al [HotQCD], Phys. Lett. B795, 15 (2019), arXiv:1812.08235

The Chiral PHASE TRANSITION in (2+1)-flavor QCD

$$M \sim m_s \frac{\partial \ln Z}{\partial m_l}$$



$$m_l = (m_u + m_d)/2 \Rightarrow 0$$

m_s fixed, physical

"magnetic" susceptibility

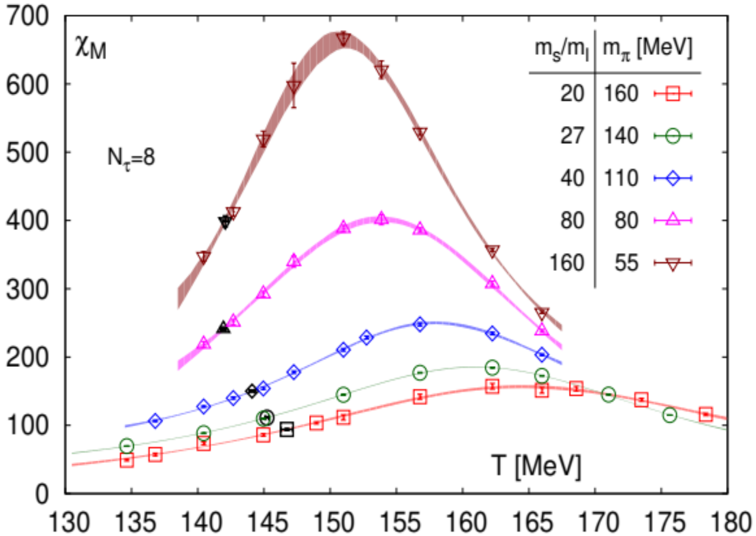
$\sim (m_s/m_l)^{0.79}$

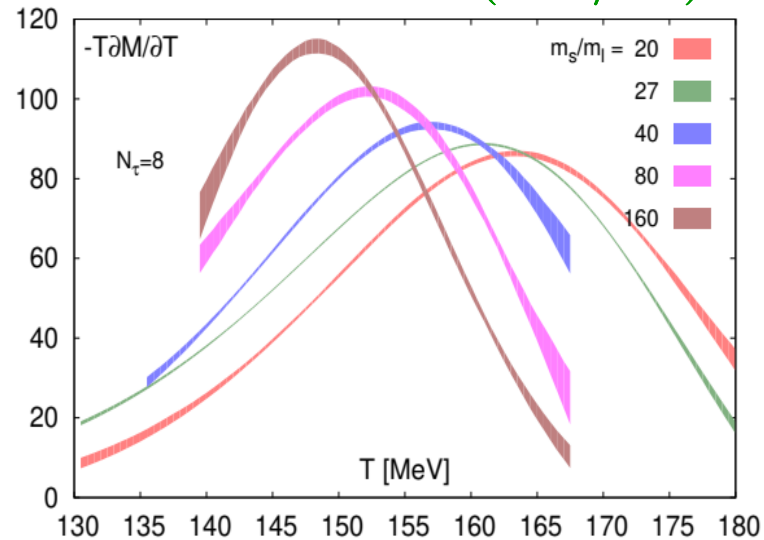
$(160/27)^{0.79} \sim 4$

"mixed" susceptibility

$\sim (m_s/m_l)^{0.34}$

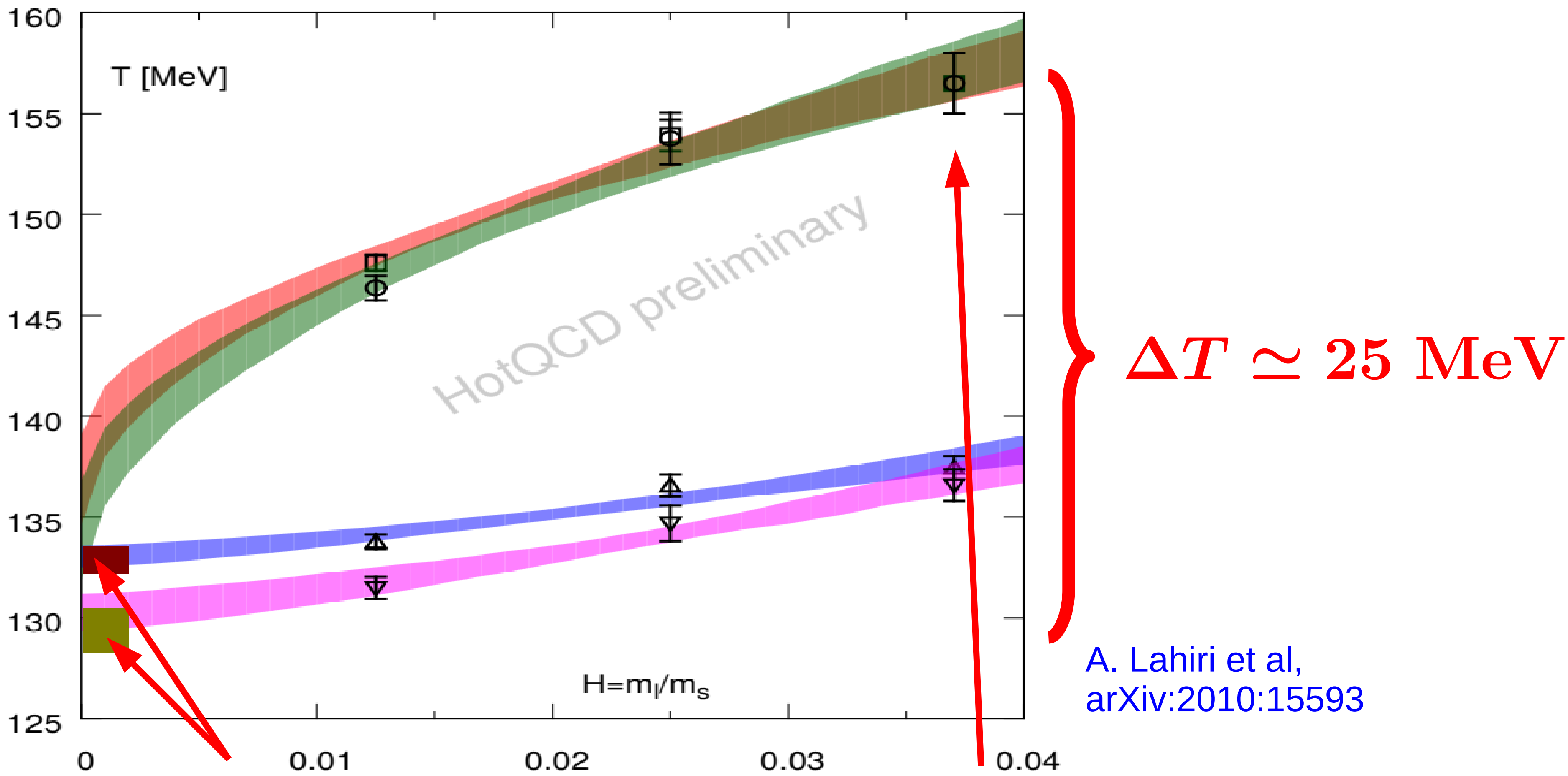
$(160/27)^{0.34} \sim 1.8$





H.T. Ding et al. (HotQCD), PRL 123 (2019) 15, arXiv:1903.04801

Chiral PHASE TRANSITION temperature



A. Lahiri et al,
arXiv:2010:15593

chiral limit extrapolations

physical masses

$$T_c^0 = 132^{+3}_{-6}\ \mathrm{MeV}$$

$$T_{pc}^{phys} = (156.5 \pm 1.5)\ \mathrm{MeV}$$

H.-T. Ding et al [HotQCD],
arXiv:1903.04801

A. Bazavov et al [HotQCD],
arXiv:1812.08235

Pseudo-critical temperature at non-zero μ_B

– μ_B-dependent shift of maxima in susceptibilities

$$\frac{\partial \chi_M(T,\mu_B)}{\partial T} = 0 \quad \text{or} \quad \frac{\partial^2 M(T,\mu_B)}{\partial T^2} = 0$$

$$\Rightarrow \quad T_{pc}(\mu_B) = T_{pc}\left(1 - \kappa_2 \left(\frac{\mu_B}{T_c}\right)^2 - \kappa_4 \left(\frac{\mu_B}{T_c}\right)^4 + ...\right)$$

– universality relations also relate derivatives with respect to T and μ_B

$$\frac{1}{VT^3} \ln Z(V,T,\vec{\mu}) \sim -h^{(2-\alpha)/\beta\delta} f_f(t/h^{1/\beta\delta})$$

$$h \sim \frac{m_q}{T_c} \;,\; t \sim \frac{T-T_c}{T_c} + \kappa_2 \left(\frac{\mu_B}{T}\right)^2 \quad \Leftrightarrow \quad \frac{\partial^2}{\partial(\mu_B/T)^2} \simeq \frac{\partial}{\partial T}$$

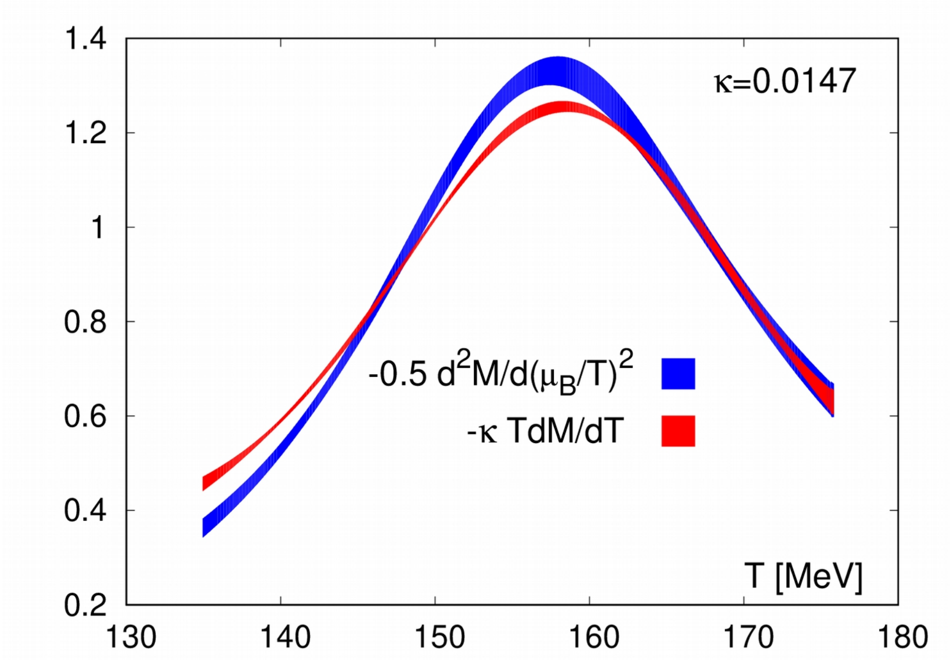
FK et al., arXiv:1009.5211

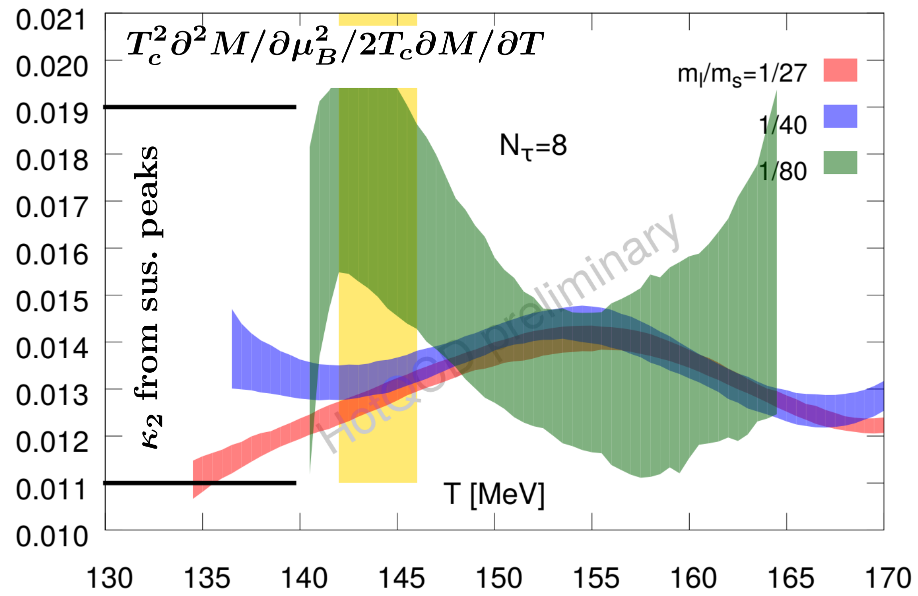
$$\Rightarrow \quad \kappa_2 \simeq \frac{T^2 \partial^2 M/\partial \mu_B^2}{2T \partial M/\partial T}$$

Critical behavior and higher order cumulants
– towards the chiral limit –

critical behavior in chiral observables: derivatives of the chiral condensate

$$M(T,\mu_B) = M(T,0) + \frac{\partial M}{\partial T}(T-T_c) + \frac{1}{2}\frac{\partial^2 M}{\partial(\mu_B/T)^2}\bigg|_{\mu_B=0}\left(\frac{\mu_B}{T}\right)^2 + \ldots$$





Mugdha Sarkar (HotQCD), Lattice 2021

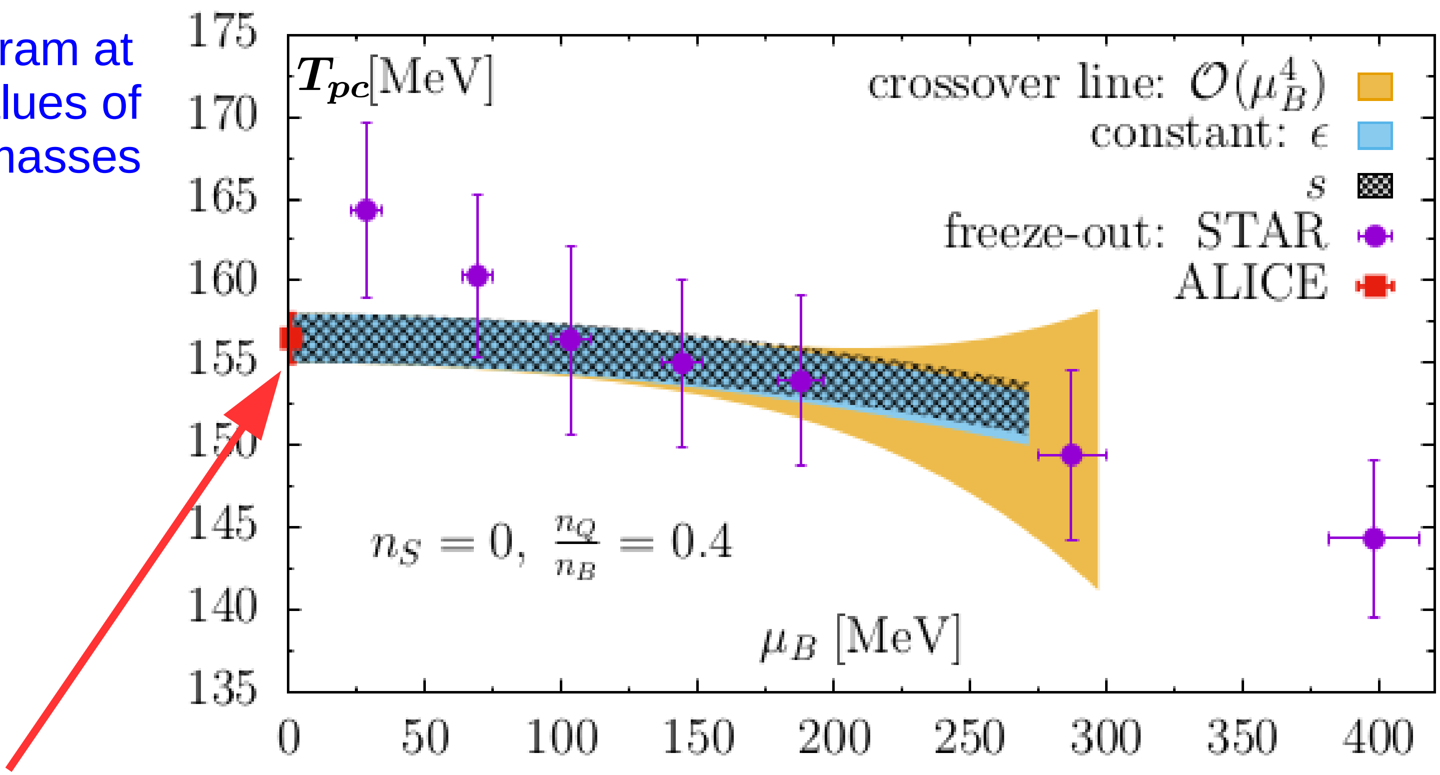
$$t \sim \frac{T-T_c}{T_c} + \kappa_2(H)\left(\frac{\mu_B}{T}\right)^2 , \; H = m_l/m_s$$

curvature of crossover line only mildly dependent on H

Phases of strong-interaction matter

$$T_{pc}(\mu_B) = T_{pc}\left(1 - \kappa_2\left(\frac{\mu_B}{T_c}\right)^2 - \kappa_4\left(\frac{\mu_B}{T_c}\right)^4 + \ldots\right)$$

phase diagram at physical values of the quark masses



STAR: arXiv:1701.07065

A. Andronic et al., Nature 561 (2018) 321

$T_{pc} = (156.5 \pm 1.5)$ MeV

$\kappa_2 = 0.012(4)$

$\kappa_4 = 0.000(4)$

A. Bazavov et al. [HotQCD], Phys. Lett. B795, 15 (2019), arXiv:1812.08235

$T_{pc} = (158.0 \pm 0.6)$ MeV

$\kappa_2 = 0.0153(18)$

$\kappa_4 = 0.00032(67)$

S. Borsanyi, et al, arXiv:2002.02821

Critical behavior and higher order cumulants
– Taylor expansion –

Taylor expansion of the **QCD** pressure: $\frac{P}{T^4} = \frac{1}{VT^3} \ln Z(T, V, \mu_B, \mu_Q, \mu_S)$

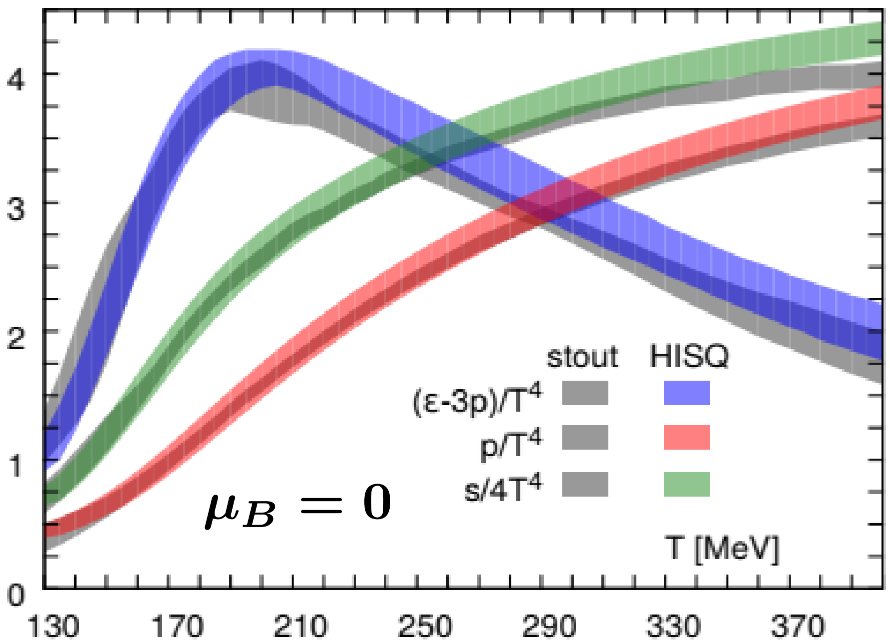
$$\frac{P}{T^4} = \sum_{i,j,k=0}^{\infty} \frac{1}{i!j!k!} \chi_{ijk}^{BQS}(T) \left(\frac{\mu_B}{T}\right)^i \left(\frac{\mu_Q}{T}\right)^j \left(\frac{\mu_S}{T}\right)^k$$

cumulants of net-charge fluctuations and correlations:

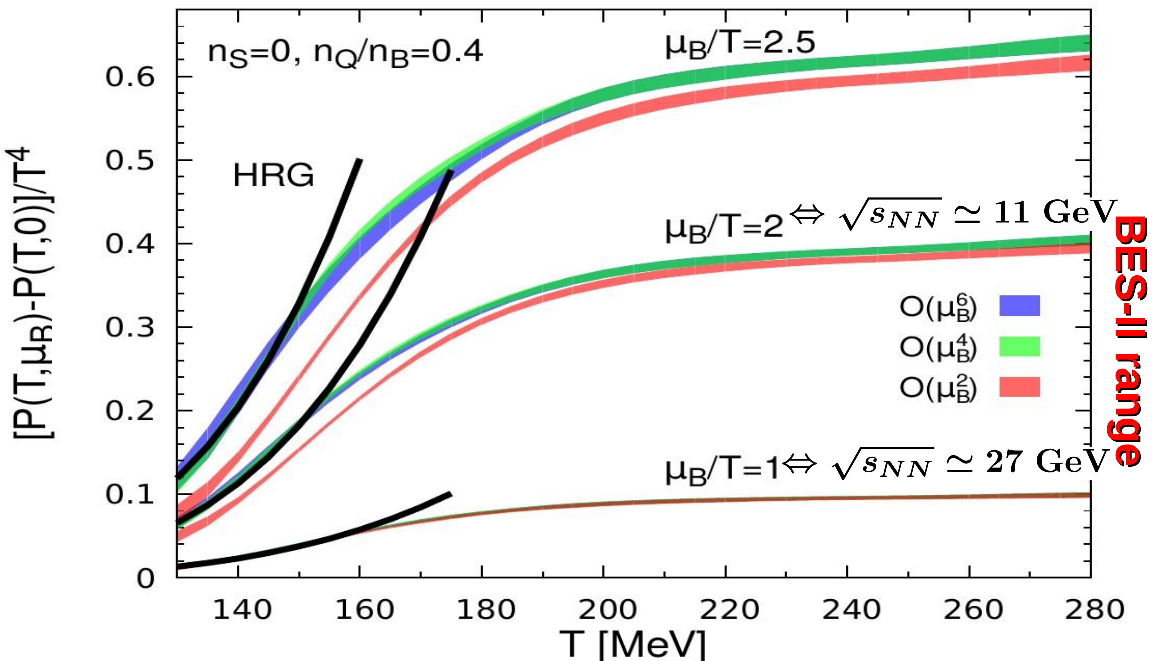
$$\chi_{ijk}^{BQS} = \left.\frac{\partial^{i+j+k} P/T^4}{\partial\hat{\mu}_B^i \partial\hat{\mu}_Q^j \partial\hat{\mu}_S^k}\right|_{\mu_{B,Q,S}=0} \quad , \quad \hat{\mu}_X \equiv \frac{\mu_X}{T}$$

Equation of state of (2+1)-flavor QCD: $\mu_B/T > 0$

$$\frac{\Delta(T,\mu_B)}{T^4} = \frac{P(T,\mu_B)-P(T,0)}{T^4} = \frac{\chi_2^B}{2}\left(\frac{\mu_B}{T}\right)^2 + \frac{\chi_4^B}{24}\left(\frac{\mu_B}{T}\right)^4 + \frac{\chi_6^B}{720}\left(\frac{\mu_B}{T}\right)^6 + \ldots$$



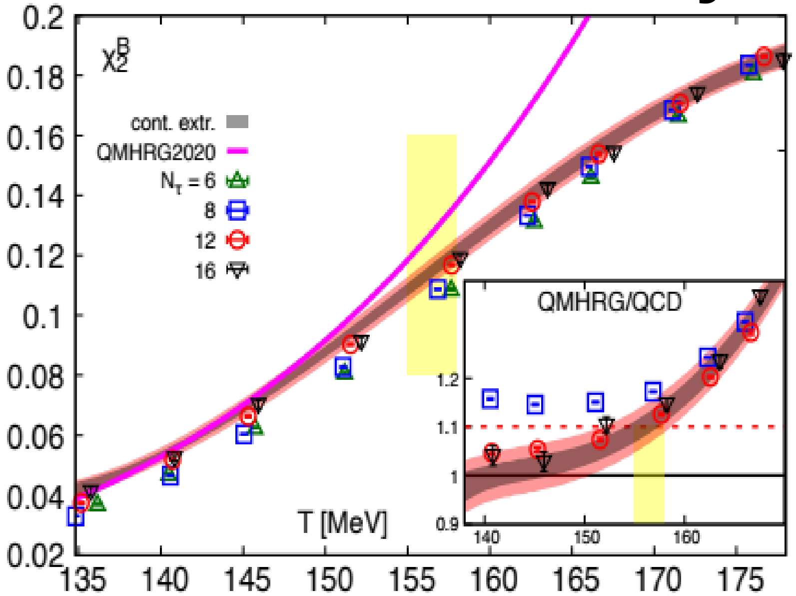
(10-30)% contribution to total pressure at $\mu_B/T = 2$

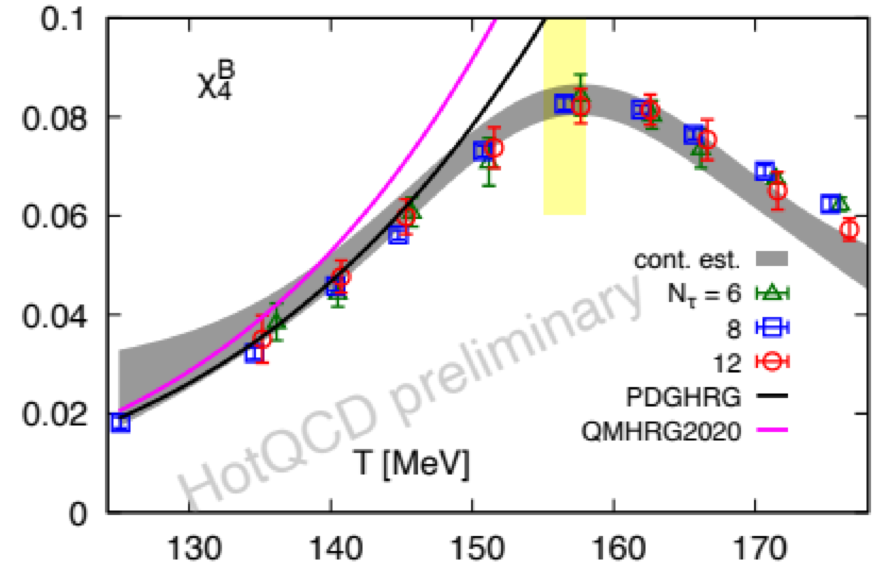


⇒ The EoS/pressure is well controlled for $\mu_B/T \le 2$ or equivalently $\sqrt{s_{NN}} \ge 11$ GeV

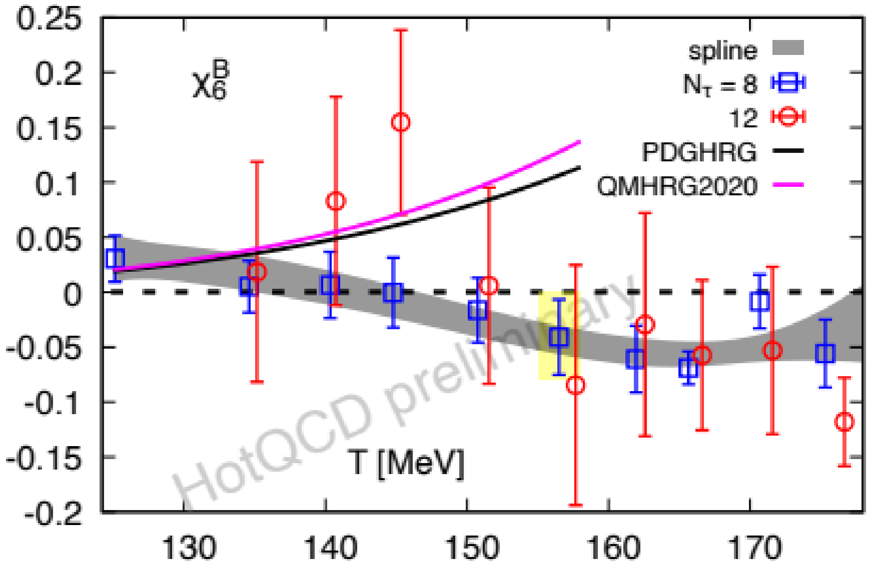
⇒ convergence of expansions for higher order derivatives increasingly worse

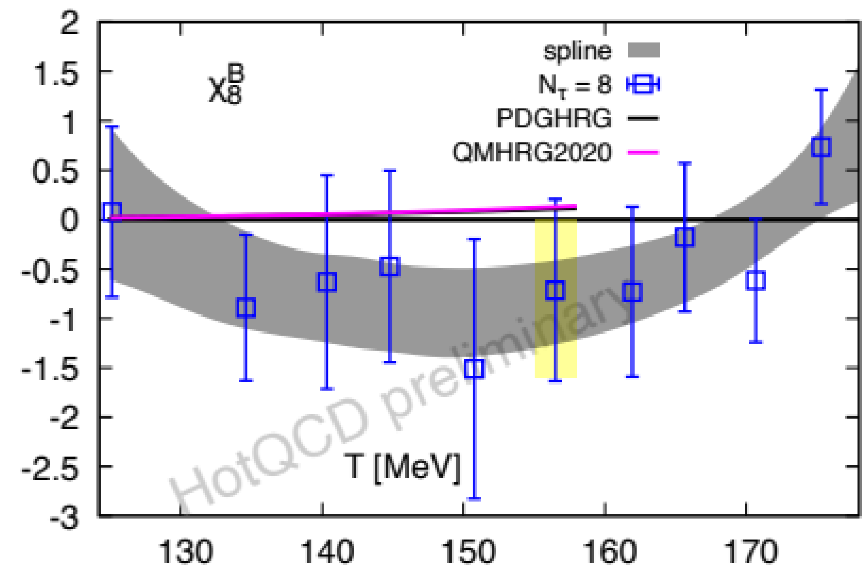
Up to 8th order cumulants are used frequently
– Taylor expansion –





D. Bollweg et al. (HotQCD), arXiv:2107.10011



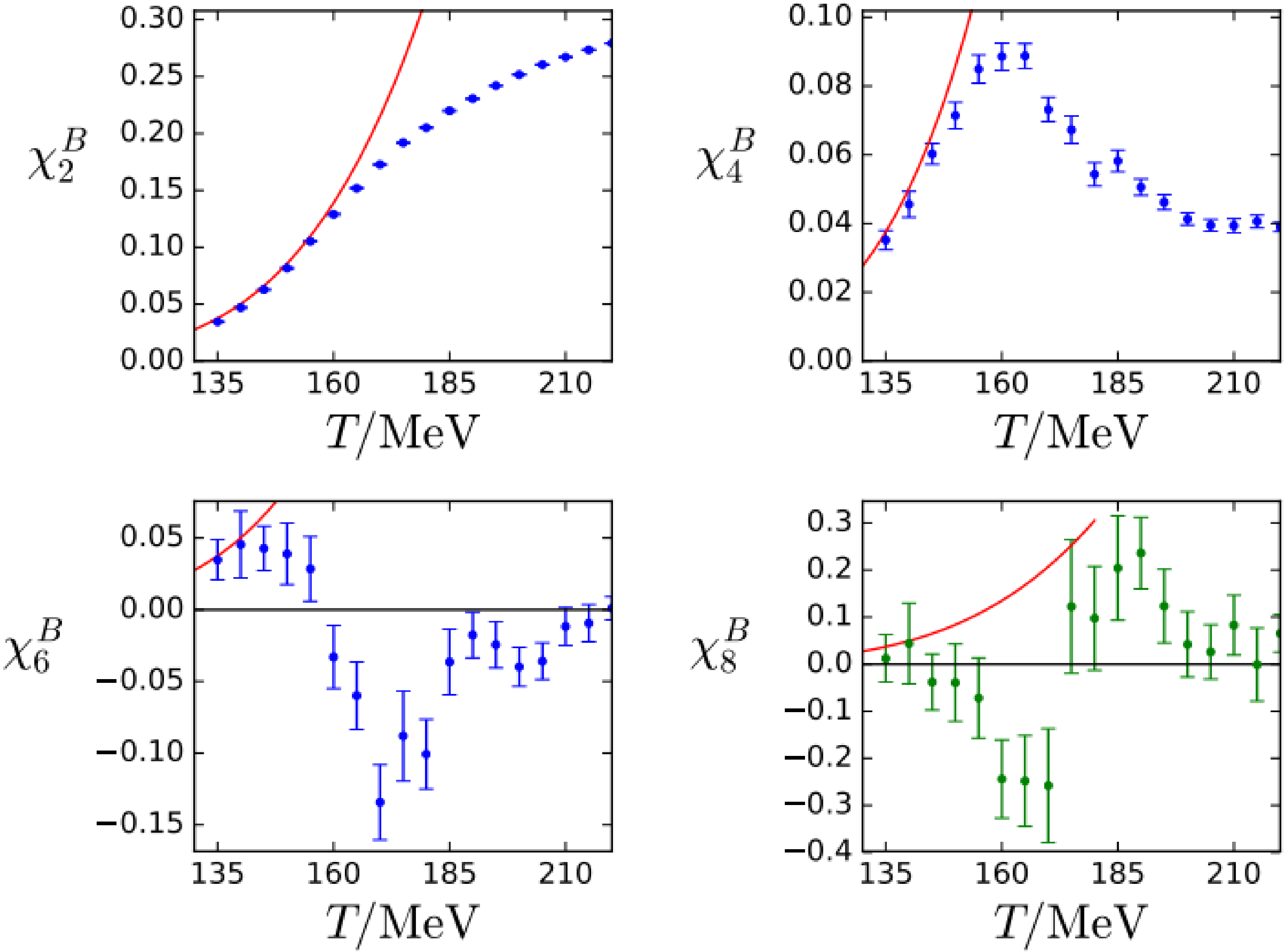


A. Bazavov et al. (HotQCD), Phys. Rev. D 101 (2020) 074502, arXiv:2001.08530

control basic features of skewness up to hyper-kurtosis ratios

Up to 8th order cumulants are used frequently
– imag. chem. pot. extrapolations –

$48^3 \times 12$



S. Borsanyi et al. , JHEP 10 (2018) 205, arXiv:1805.04445

Cumulant ratios on the pseudo-critical line

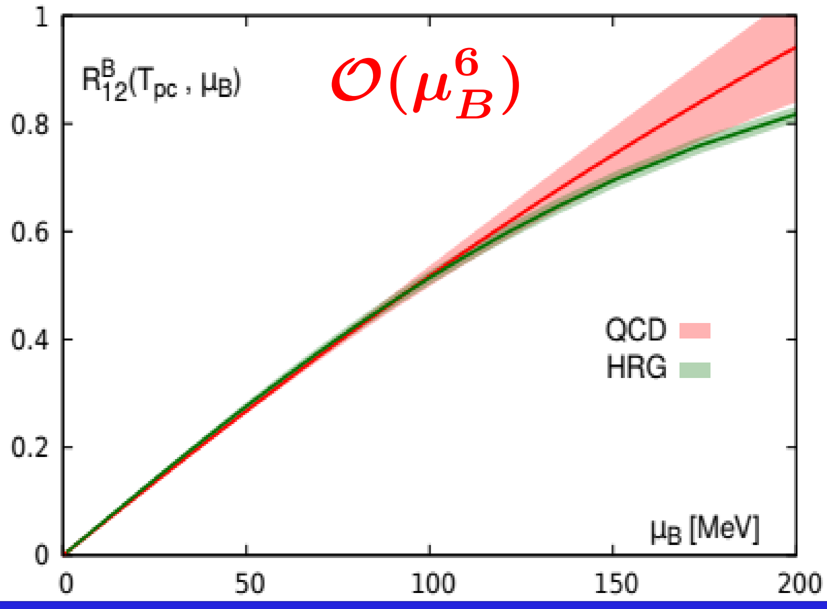
A. Bazavov et al. (HotQCD), PRD 101, 074502 (2020)
arXiv:2001.08530

$$n_S = 0,\ n_Q/n_B = 0.4:$$

$$R^B_{nm} = \frac{\chi^B_n(T,\mu_B)}{\chi^B_m(T,\mu_B)} = \frac{\sum_{k=1}^{k_{max}} \tilde{\chi}^{B,k}_n(T)\hat{\mu}^k_B}{\sum_{l=1}^{l_{max}} \tilde{\chi}^{B,l}_m(T)\hat{\mu}^l_B}$$

mean over variance

$$R^X_{12} = (M/\sigma^2)_X = \frac{\chi^X_1}{\chi^X_2}$$



$\mathcal{O}(\mu_B^4)$

skewness

$$R^X_{31} = (\frac{S\sigma^3}{M})_X = \frac{\chi^X_3}{\chi^X_1}$$

kurtosis

$$R^X_{42} = (\kappa\sigma^2)_X = \frac{\chi^X_4}{\chi^X_2}$$

$\mathcal{O}(\mu_B^2)$

hyper-skewness

$$R^X_{51} = (\frac{S^H\sigma^5}{M})_X = \frac{\chi^X_5}{\chi^X_1}$$

hyper-kurtosis

$$R^X_{62} = (\kappa^H\sigma^4)_X = \frac{\chi^X_6}{\chi^X_2}$$

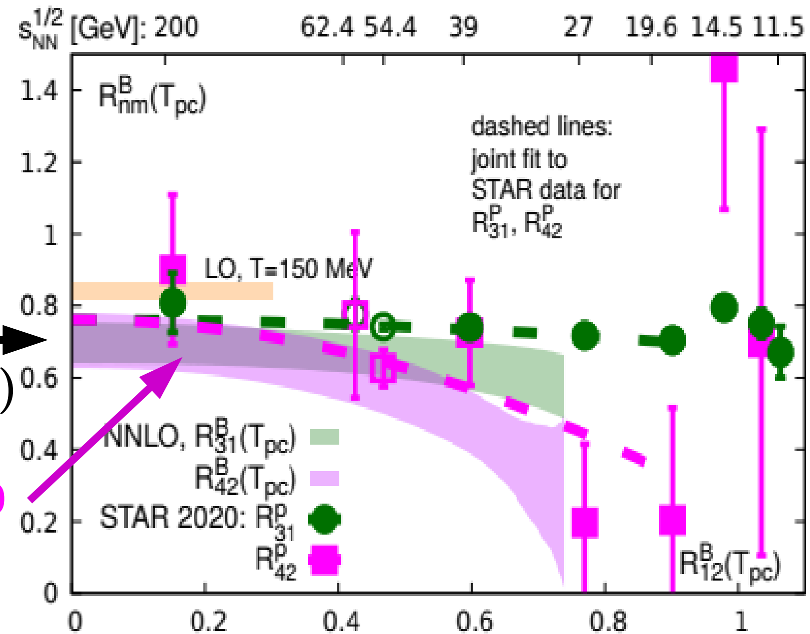
X=B (QCD), P (STAR)

Skewness, kurtosis, **hyper-skewness and hyper-kurtosis** ratios on the pseudo-critical line

A. Bazavov et al. (HotQCD), arXiv:2001.08530

new STAR data:
STAR, arXiv:2105.14698

$$R_{nm}^B = \frac{\chi_n^B(T,\mu_B)}{\chi_m^B(T,\mu_B)} = \frac{\sum_{k=1}^{k_{max}} \tilde{\chi}_n^{B,k}(T)\hat{\mu}_B^k}{\sum_{l=1}^{l_{max}} \tilde{\chi}_m^{B,l}(T)\hat{\mu}_B^l}$$

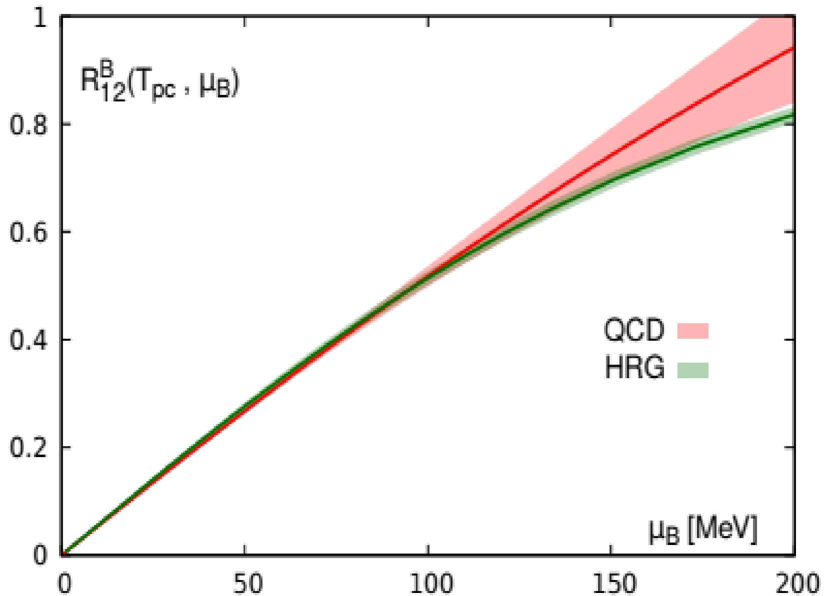


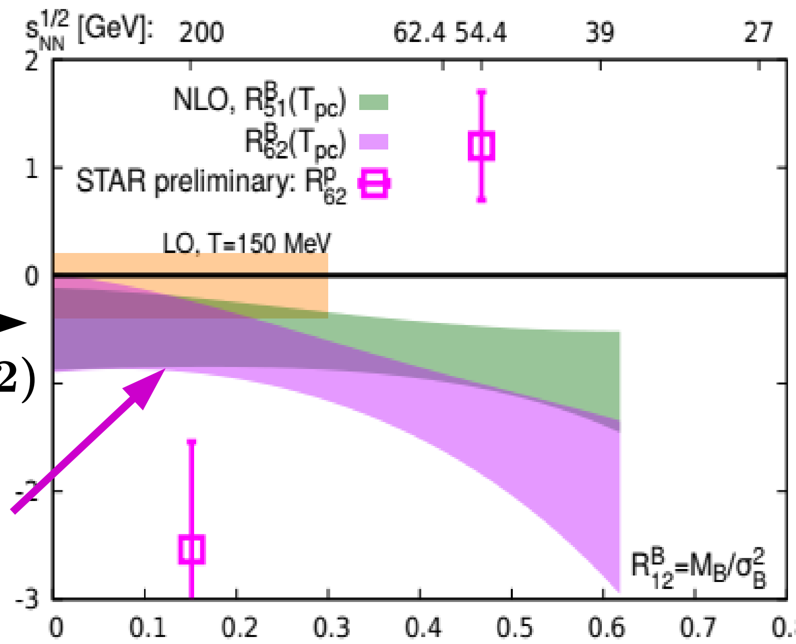
$$\frac{\chi_4^B}{\chi_2^B} = 0.69(3)$$

$\chi_6^B(T_{pc}) < 0$

convert μ_B/T to M_B/σ_B^2

(avoids model dependent determination of μ_B/T)



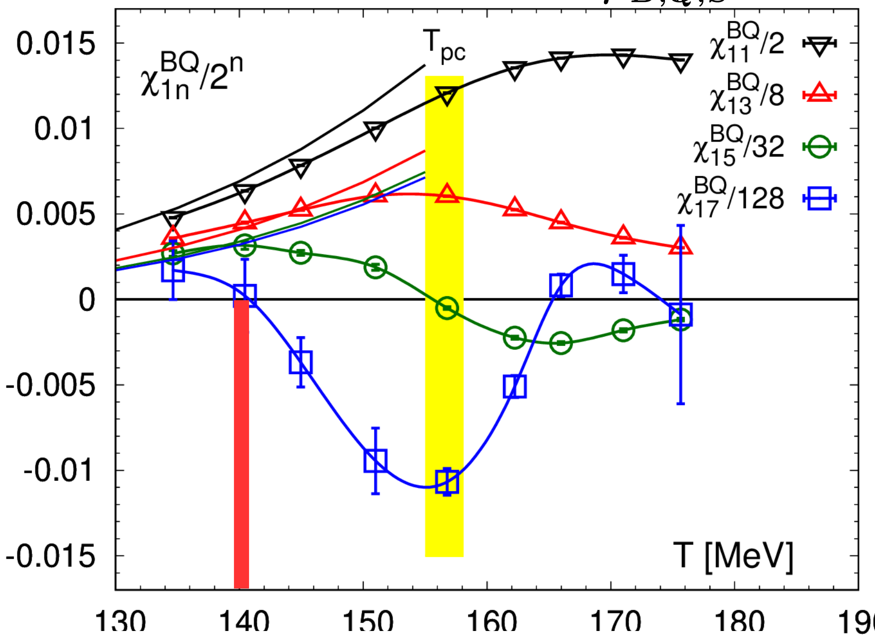


$$\frac{\chi_6^B}{\chi_2^B} = -0.3(2)$$

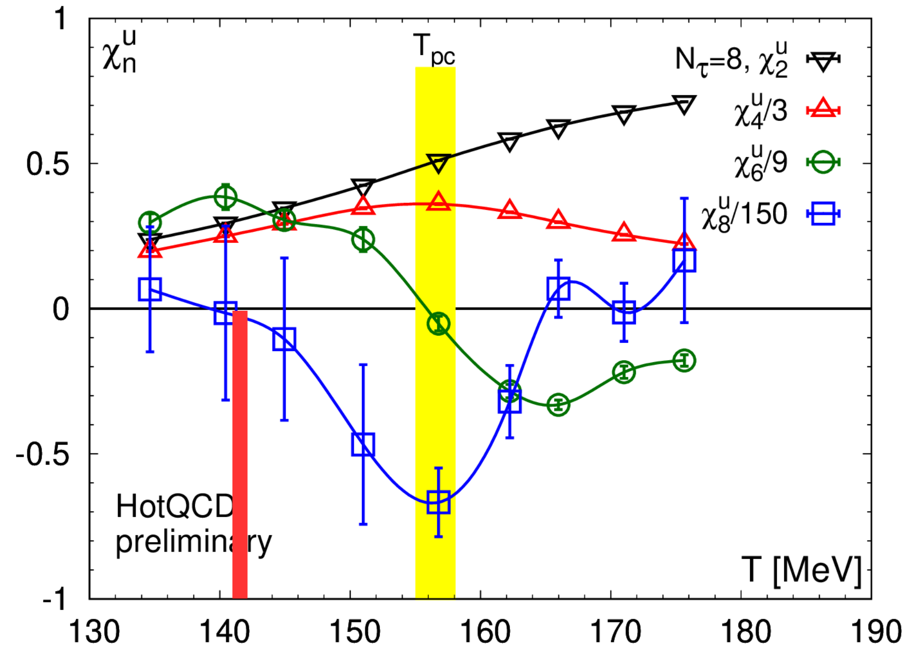
$\chi_8^B(T_{pc}) < 0$

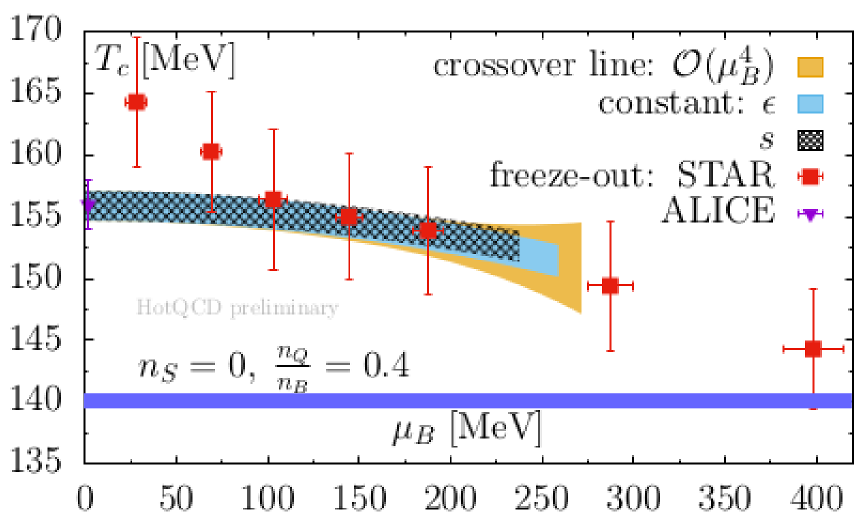
Critical behavior and higher order cumulants

$$\chi_{1n}^{BQ} = \left. \frac{\partial^{n+1} P/T^4}{\partial \hat{\mu}_B \partial \hat{\mu}_Q^n} \right|_{\mu_{B,Q,S}=0}$$



$$\chi_n^u = \left. \frac{\partial^n P/T^4}{\partial \hat{\mu}_u^n} \right|_{\mu_{u,d,s}=0}$$





many 8th order cumulants turn negative for

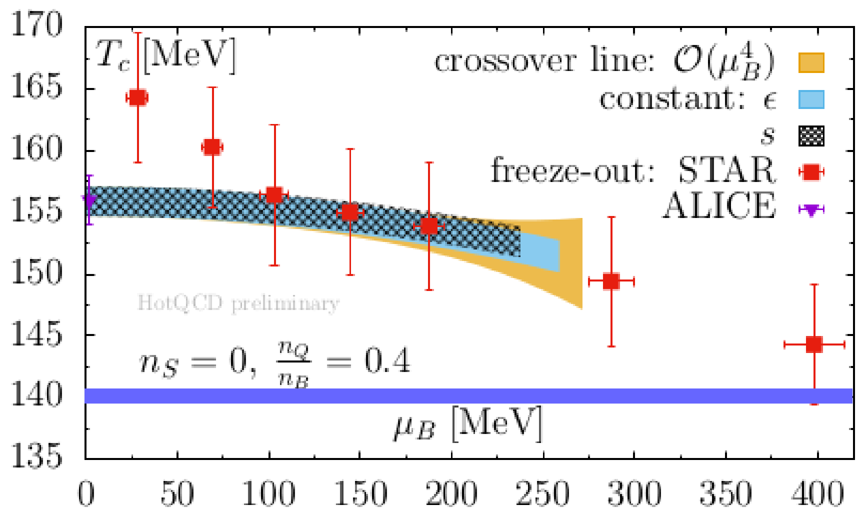
$$T^- \gtrsim (140 - 145)\ \mathrm{MeV}$$

zeroes in the complex μ_B limit radius of convergence

→ plausible scenario:

$$T_{CEP} < 140\mathrm{MeV}\ ,\ \mu_B^{CEP} > 400\mathrm{MeV}$$

Critical behavior and higher order cumulants



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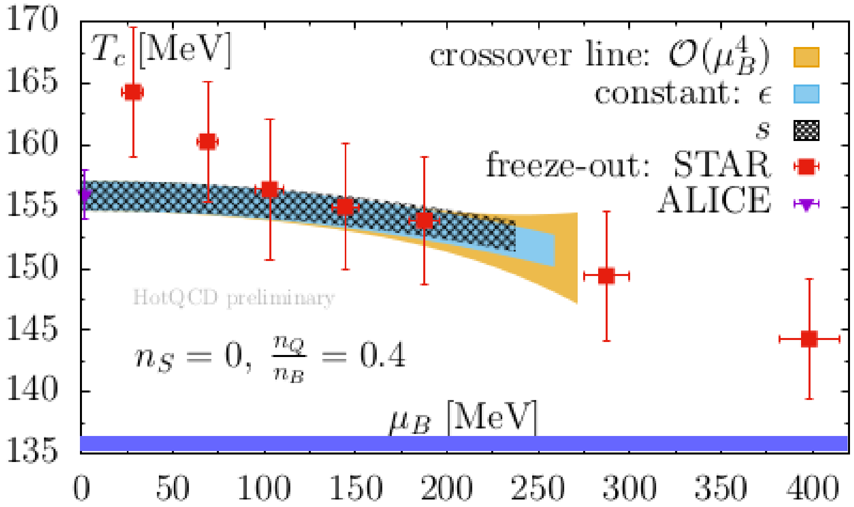
– higher order cumulants will continue to "oscillate" at even lower T, if convergence of Taylor series is limited by a complex zero

$$T_{CEP} < 140\mathrm{MeV}\ ,\ \mu_B^{CEP} > 400\mathrm{MeV}$$

Critical behavior and higher order cumulants



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$$T_{CEP} < 140\mathrm{MeV}\ ,\ \mu_B^{CEP} > 400\mathrm{MeV}$$

Lee-Yang edge zeroes

Exploit analytic structure of scaling functions:

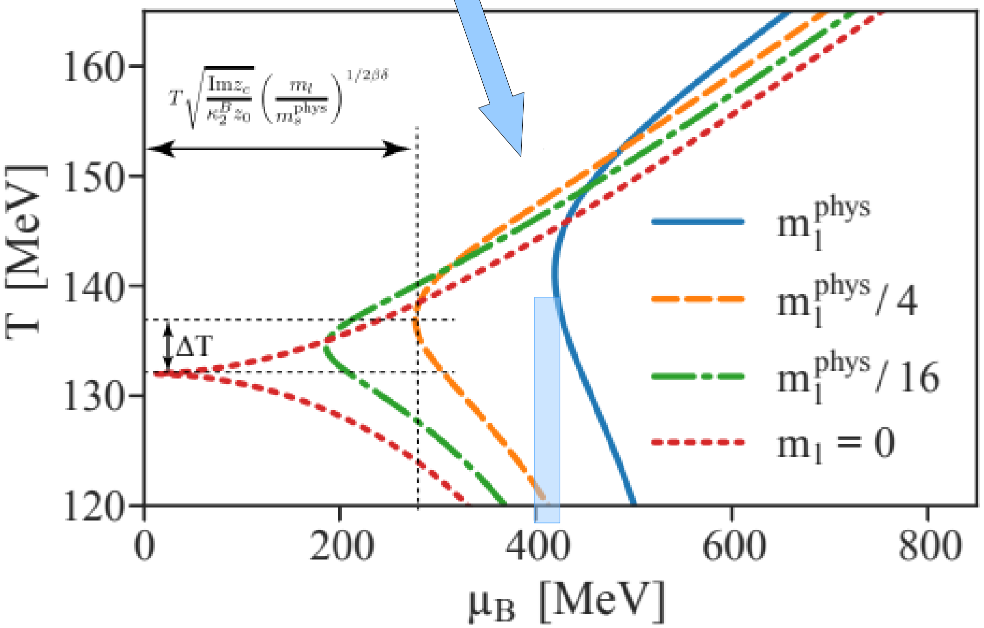
M. A. Stephanov, hep-lat/0603014

– using input on non-universal parameters (T_c, κ_2, z_0) allows to estimate the radius of convergence deduced from universal properties of O(4) scaling function

S. Mukherjee, V. Skokov, arXiv: 1909.04639

– use imag-μ_B simulations to reach complex Lee-Yang edge zeroes

C. Schmidt et al.,arXiv:2101.02254



→ **Taylor expansions will not allow to reach the CEP, if $\mu_B^{CEP} > 400$ MeV**

→ **Taylor series needs to be resummed**

Resumming Taylor series

– conformal mappings — V. Skokov, K. Morita, B. Friman, arXiv:1008.4549
M. Giordano et al., arXiv:2004.10800
G. Basar, arXiv:2105.08080

– partial resummation — S. Modal, S. Mukherjee, P. Hegde , arXiv:2106.03165

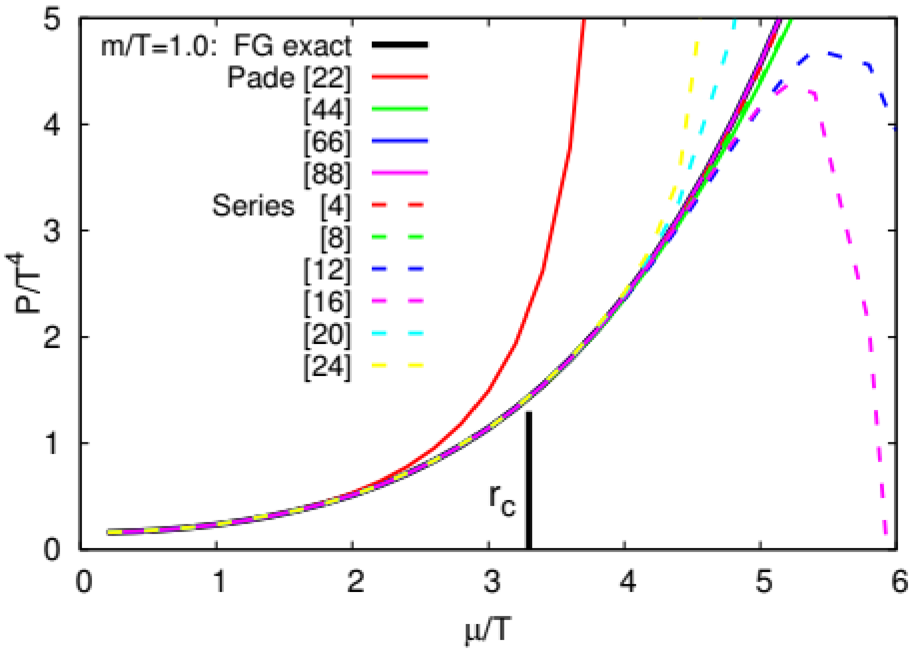
– **Pade resummation**

a simple example:

– Taylor series for a relativistic Fermi gas as function of chemical potential
– radius of convergence controlled by an imaginary zero at $\mu_c/T = i\pi$
– series expansions in real μ break down at μ_c

– diagonal Pades, P[nn], have no problem avoiding this singularity

– phase transitions are signaled by (stable) zeroes in Pade approximants



Resumming Taylor series

$$c_n = \chi_n^B / n!$$
$$c_{n2} = c_n / c_2$$

Pade resummation

$$P(T,\mu_B)/T^4 = P(T,0)/T^4 + P_2(T,x)\ ,\ x = \mu_B/T$$

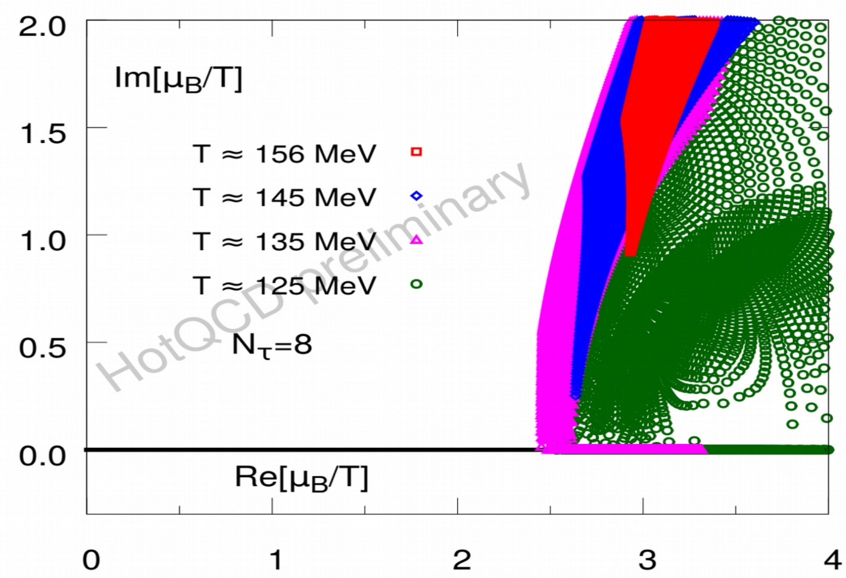
$$P_2(T,x) = c_2(T)x^2 + c_4(T)x^4 + c_6(T)x^6 + c_8(T)x^8$$

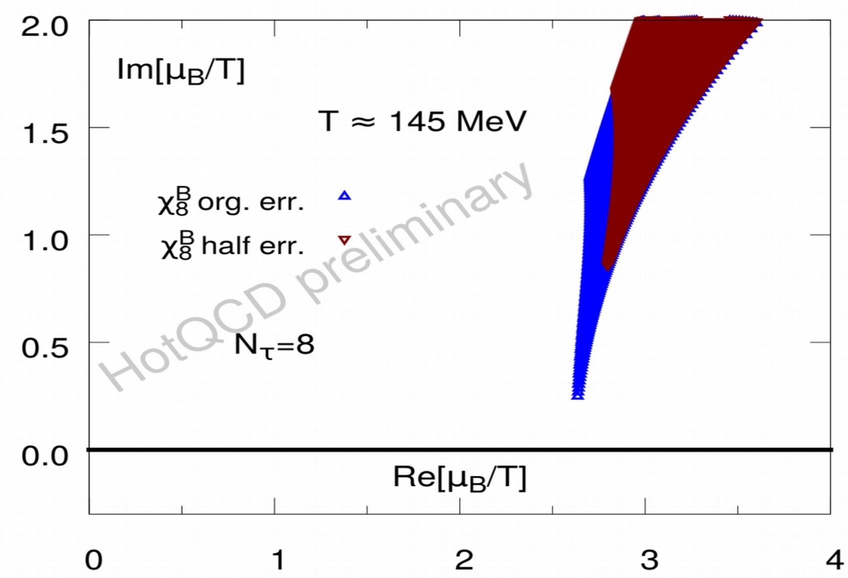
$$P[4,4](x) = c_2 \frac{(c_{42}^2 - c_{62})x^2 + \left(c_{42}^3 - 2c_{42}c_{62} + c_{82}\right)x^4}{(c_{42}^2 - c_{62}) + (c_{82} - c_{42}c_{62})x^2 + (c_{62}^2 - c_{42}c_{82})x^4}$$

HotQCD preliminary: thanks to Jishnu Goswami, Anirban Lahiri,...

– possible location of (positive) pole of the [4,4] Pade within current errors on

$$c_6 = \chi_6^B/720,\ c_8 = \chi_8^B/40320$$





Resumming Taylor series

$$c_n = \chi_n^B/n!$$
$$c_{n2} = c_n/c_2$$

Pade resummation

HotQCD preliminary:
thanks to
Jishnu Goswami,
Anirban Lahiri,...

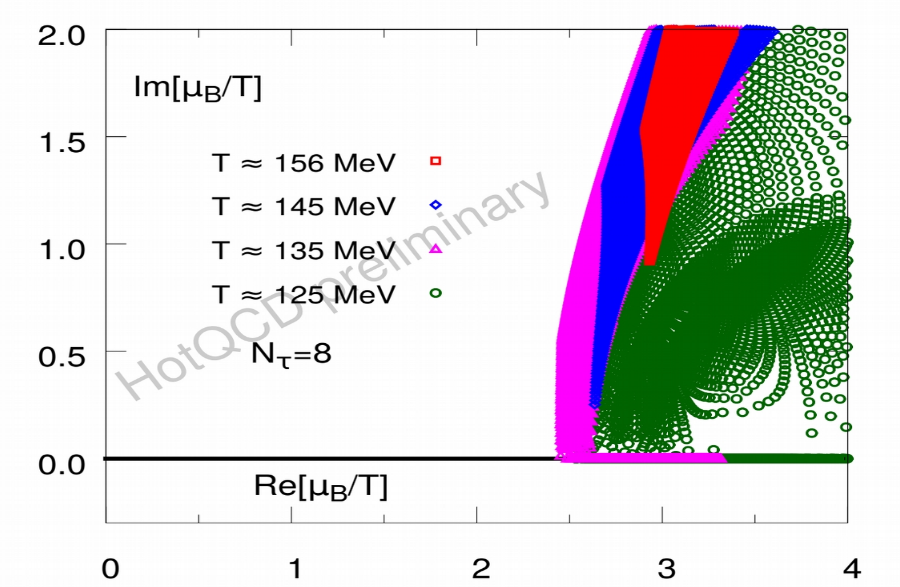
$$P(T,\mu_B)/T^4 = P(T,0)/T^4 + P_2(T,x)\ ,\ x=\mu_B/T$$

$$P_2(T,x) = c_2(T)x^2 + c_4(T)x^4 + c_6(T)x^6 + c_8(T)x^8$$

$$P[4,4](x) = c_2 \frac{(c_{42}^2 - c_{62})x^2 + \left(c_{42}^3 - 2c_{42}c_{62} + c_{82}\right)x^4}{(c_{42}^2 - c_{62}) + (c_{82} - c_{42}c_{62})x^2 + (c_{62}^2 - c_{42}c_{82})x^4}$$

– possible location of (positive) pole of the [4,4] Pade within current errors on

$$c_6 = \chi_6^B/720,\ c_8 = \chi_8^B/40320$$



within current errors poles on the real axis (critical point) are possible only for

$$T \le 135\text{MeV}\ ,\ \mu_B/T > 2.5$$

consistent with:
M.Giordano et al., arXiv:2004.10800

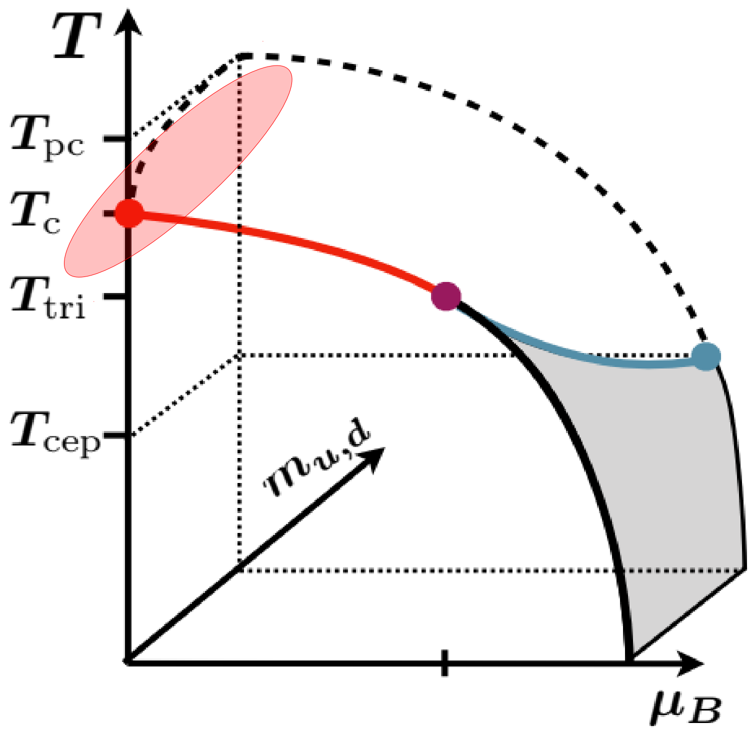
higher statistics will sharpen the constraint

Conclusion on "Critical behavior in QCD"

- close to the chiral limit thermodynamics in the vicinity of the QCD transition(s) is controlled by a universal scaling function

singular regular

$$\frac{p}{T^4} = \frac{1}{VT^3} \ln Z(V,T,\vec{\mu}) = -h^{(2-\alpha)/\beta\delta} f_f(t/h^{1/\beta\delta}) - f_r(V,T,\vec{\mu})$$



What we learned so far about the CEP in QCD from lattice QCD calculations:

I) the critical temperature is below Tc=132 MeV

II) the corresponding critical chemical potential is likely to be above 400 MeV

→ Taylor expansions need to be resummed in order to reach CEP

Question by the organizers: How far can we go?

- an attempt to give an history motivated answer

40 years of lattice QCD thermodynamics:

- an attempt to give an history motivated answer

40 years of lattice QCD thermodynamics:

the first direct evidence for the existence of a thermal phase transition in SU(Nc) gauge theories from lattice calculations has been presented at a conference in Bielefeld

Statistical Mechanics of Quarks and Hadrons
ZIF, Bielefeld, August 1980 (organizer H. Satz)

B. Svetitsky and L. McLerran, PLB 98 (1981)
J. Kuti, J. Polonyi and K. Szlachanyi PLB 98 (1981)

2020/21: speed increased by **10^{12}**, i.e. a factor 2 every year

1980/81:
first lattice calculations of the EoS

J. Engels, FK, I. Montvay, H. Satz, PLB 101 (1981)

- an attempt to give an history motivated answer

40 years of lattice QCD thermodynamics:

– the radius of convergence is not at all a limit for CEP searches based on Taylor expansions

– higher order Taylor coefficients provide crucial information in constraining the location of a CEP also outside the radius of convergence

– every 5 years computing speed increases by an order of magnitude (+ algorithmic advances)

⇨ every (4-5) years the calculation of another order in the Taylor series becomes affordable

⇨ this will lead to more stringent bounds on the possible existence and location of a

CEP in the QCD phase diagram