Unit 3
Multipolar expansion of magnetic field

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All the units will use International System (meter, kilo, second, ampere) unless specified

CONTENTS

- Definition of field harmonics
- Field harmonics of a current line
- Validity limits of field harmonics
- Some applications of tough mathematics to magnetic field measurements
- Hints on beam dynamics requirements on field harmonics

FIELD HARMONICS: MAXWELL EQUATIONS

- Maxwell equations for magnetic field

$$\nabla \cdot B = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \qquad \nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$$

- In absence of charge and magnetized material (inside a magnet)

$$\nabla \times B = \left(\frac{\partial B_y}{\partial z} - \frac{\partial B_z}{\partial y}, \frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z}, \frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x} \right) = 0$$

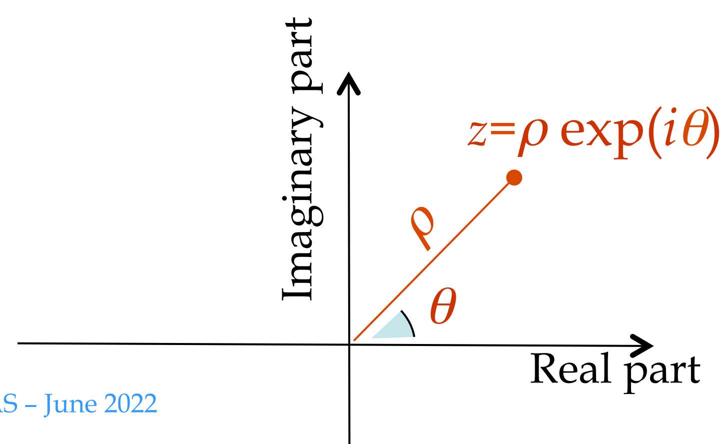
James Clerk Maxwell, Scottish (13 June 1831 – 5 November 1879)

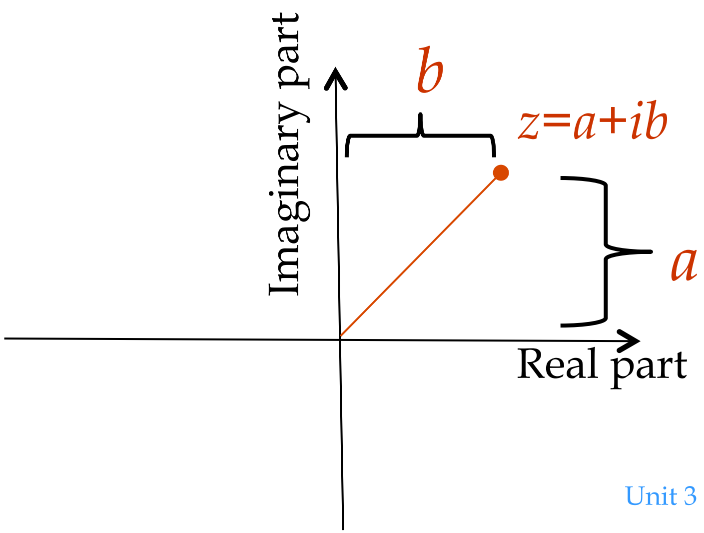
- If $\frac{\partial B_z}{\partial z} = 0$ (constant longitudinal field), then

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0 \qquad \frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x} = 0$$

  - Remember: *x* and *y* perpendicular to the beam (transverse coordinates), *z* along the beam (*s* in previous unit)

- Complex numbers are defined to be able to find a solution to the square root of negative numbers
  - This is an extension of real numbers
  - The advantage is that every equation of n^{th} degree has n solutions in the complex domain
- The trick is to define $i=\sqrt{(-1)}$
- A complex number has two components $z=a+ib$
  - Can be written also in the exponential form $z=\rho \exp(i\theta)$
    - $\rho=\sqrt{(a^2+b^2)}$





FIELD HARMONICS: ANALYTIC FUNCTIONS

- A complex function of complex variables is analytic if it coincides with its power series

$$f(z)=\sum_{n=1}^{\infty} C_n z^{n-1} \qquad f_x(x,y)+if_y(x,y)=\sum_{n=1}^{\infty} C_n (x+iy)^{n-1} \qquad (x,y)\in D$$

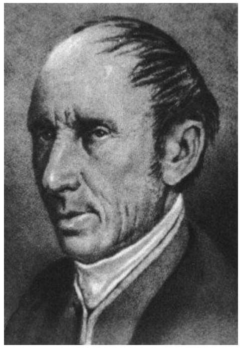
on a domain D !

  - Note: not every combination of f_x+if_y is analytic !
  - Note: domains are usually a painful part, we talk about it later

- A necessary and sufficient condition for (f_x, f_y) to have f_x+if_y analytic is that

$$\begin{cases} \dfrac{\partial f_x}{\partial x}-\dfrac{\partial f_y}{\partial y}=0 \\ \dfrac{\partial f_x}{\partial y}+\dfrac{\partial f_y}{\partial x}=0 \end{cases}$$

called the Cauchy-Riemann conditions



Augustin Louis Cauchy
French
(August 21, 1789 – May 23, 1857)

DEFINITION OF FIELD HARMONICS

- If $\frac{\partial B_z}{\partial z} = 0$

  Maxwell gives
  $$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = 0$$
  $$\frac{\partial B_y}{\partial y} + \frac{\partial B_x}{\partial x} = 0$$

$$\begin{cases} \frac{\partial f_x}{\partial x} - \frac{\partial f_y}{\partial y} = 0 \\ \frac{\partial f_x}{\partial y} + \frac{\partial f_y}{\partial x} = 0 \end{cases}$$

Georg Friedrich Bernhard Riemann, German (November 17, 1826 - July 20, 1866)

and therefore the function B_y+iB_x is analytic

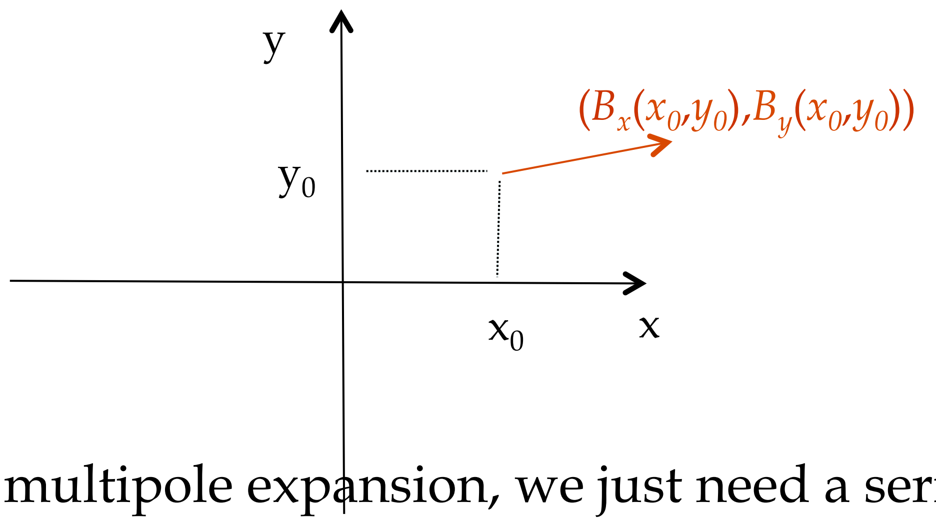
$$B_y(x,y) + iB_x(x,y) = \sum_{n=1}^{\infty} C_n (x+iy)^{n-1} \qquad (x,y) \in D$$

where C_n are complex coefficients

Please note the analytic function is not B_x+iB_y but B_y+iB_x … in other books you can find B_x-iB_y , it is the same

DEFINITION OF FIELD HARMONICS

- To describe the magnetic field we associate to each point of the space a vector
  - For instance to store in a computer we need to create a matrix (grid in the space) and for every point of the grid we store two numbers

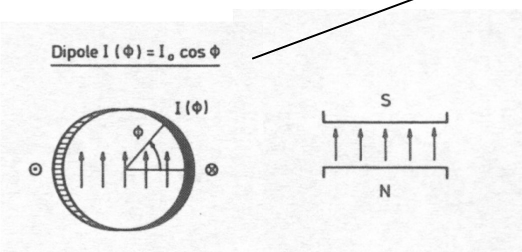


- Thanks to the multipole expansion, we just need a series of complex coefficient
  - A matematician would say "we reduce the description of a function from R^2 to R^2 to a (simple) series of complex coefficients"
- Attention !! We lose something (the function outside *D*) (we will come back to this)

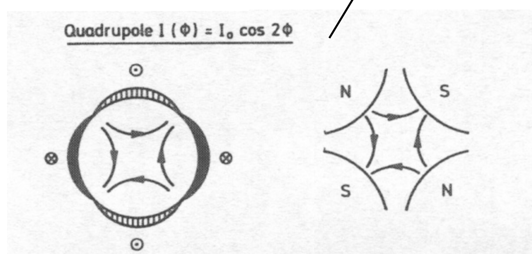
DEFINITION OF FIELD HARMONICS

$$B_y(x,y) + iB_x(x,y) = \sum_{n=1}^{\infty} C_n \left(x+iy\right)^{n-1} = C_1 + C_2(x+iy) + \ldots \qquad (x,y) \in D$$

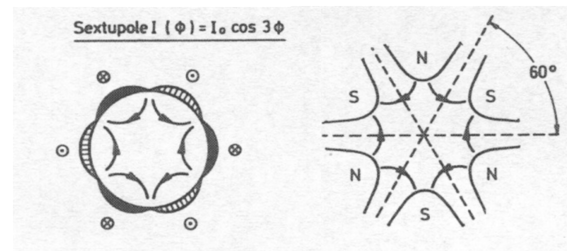
- Each coefficient corresponds to a "pure" multipolar field



A dipole



A quadrupole
[from P. Schmuser et al, pg. 50]



A sextupole

- Magnets usually aim at generating a single multipole
  - Dipole, quadrupole, sextupole, octupole, decapole, dodecapole …
  - Combined magnets: provide more components at the same time (for instance dipole and quadrupole) – more common in low energy rings, resistive magnets – one superconducting example: JPARC (KEK, Japan)

DEFINITION OF FIELD HARMONICS

$$B_y(x,y)+iB_x(x,y)=\sum_{n=1}^{\infty}C_n\left(x+iy\right)^{n-1}=C_1+C_2(x+iy)+\ldots \qquad (x,y)\in D$$

- The field can be described by in infinite number of complex coefficients
  - This means a double infinite number of real numbers
- We made a huge simplification in the description of the magnetic field
- But the story is not ended …
  - The series is converging quite rapidly so if you stop at order ten the precision is good enough – only 20 numbers – we will see this in this unit
- For the optimization of the layout of a magnet, there are symmetries that set to zero many of these multipoles
  - An optimization problem (coil layout in blocks) involves only about 5 numbers !

DEFINITION OF FIELD HARMONICS

$$B_y(x,y) + iB_x(x,y) = \sum_{n=1}^{\infty} C_n (x+iy)^{n-1} = \sum_{n=1}^{\infty} (B_n + iA_n)(x+iy)^{n-1}$$

- The field harmonics are rewritten as

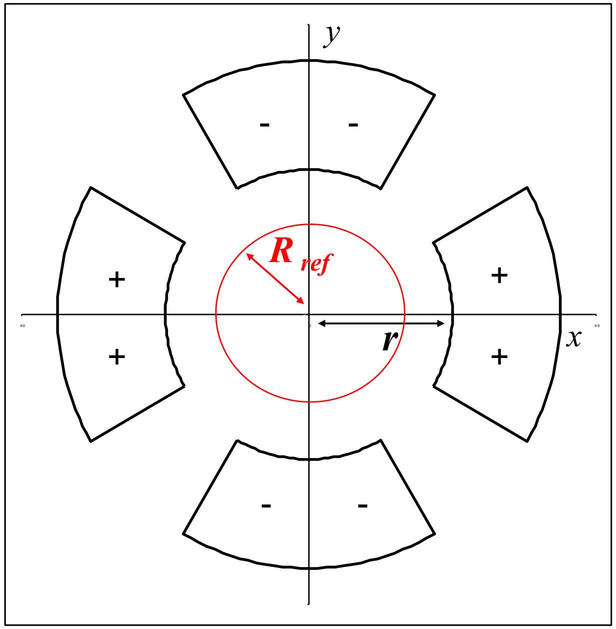
$$B_y + iB_x = 10^{-4} B_1 \sum_{n=1}^{\infty} (b_n + ia_n) \left(\frac{x+iy}{R_{ref}} \right)^{n-1}$$

  - We factorize the main component (B_1 for dipoles, B_2 for quadrupoles)
  - We introduce a reference radius R_{ref} to have dimensionless coefficients
  - We factorize 10^{-4} since the deviations from ideal field in superconducting magnets for particle accelerators have to be ~0.01%
- The coefficients b_n, a_n are called normalized multipoles
  - b_n are the normal, a_n are the skew (adimensional)

DEFINITION OF FIELD HARMONICS

$$B_y + iB_x = 10^{-4} B_1 \sum_{n=1}^{\infty} (b_n + ia_n) \left(\frac{x + iy}{R_{ref}} \right)^{n-1}$$

- Reference radius is usually chosen as 2/3 of the aperture radius
  - This is done to have numbers for the multipoles that are not too far from 1
- Some wrong ideas about reference radius
  - Wrong statement 1: "the expansion is valid up to the reference radius"
    - The reference radius has no physical meaning, it is as choosing meters of mm
    - We will come back on the validity limit
  - Wrong statement 2: "the expansion is done around the reference radius"
    - A power series is around a point, not around a circle. The expansion is around the origin



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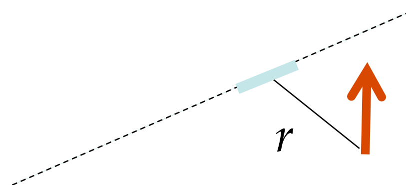
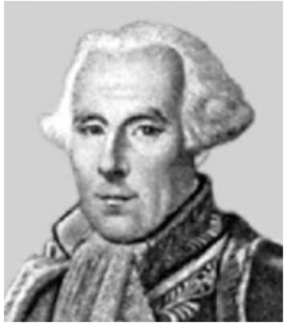
FIELD OF A CURRENT LINE

- Field given by a current line (Biot-Savart law)
  - Differential form (international system)

$$d\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{I} \times d\vec{r}}{|r|^3}$$

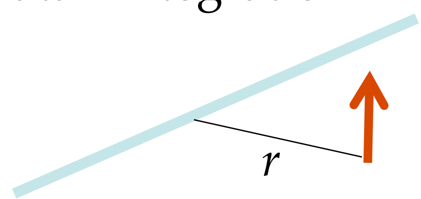




Félix Savart,
French
(June 30, 1791-March 16, 1841)

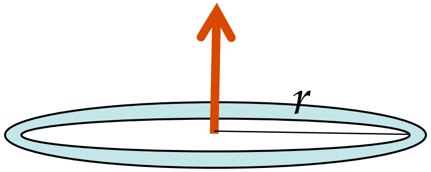
  - Infinite current line
    - A factor two is given by the atan integration

$$\vec{B}(\vec{r}) = \frac{\mu_0}{2\pi} \frac{\vec{I} \times \vec{r}}{|r|^2}$$



  - Field in a centre of a circular loop, radius r

$$B = \frac{\mu_0 I}{2r}$$

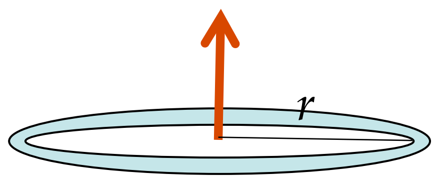


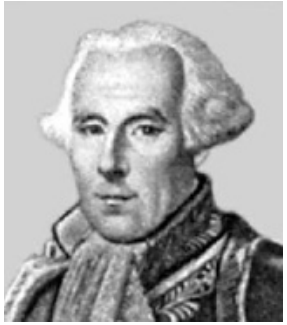
Jean-Baptiste Biot,
French
(April 21, 1774 – February 3, 1862)

ABOUT THE CONSTANT MU ZERO

- Field given by a loop – prototype of a solenoid

$$B = \frac{\mu_0 I}{2r}$$





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- About the constant μ_0
- It is terribly small … $\mu_0 = 4\pi \times 10^{-7}$ Tm/A
- This means that with 1 A at 1 m you get less than 1 μT
- This is why to make few T you need MA turns

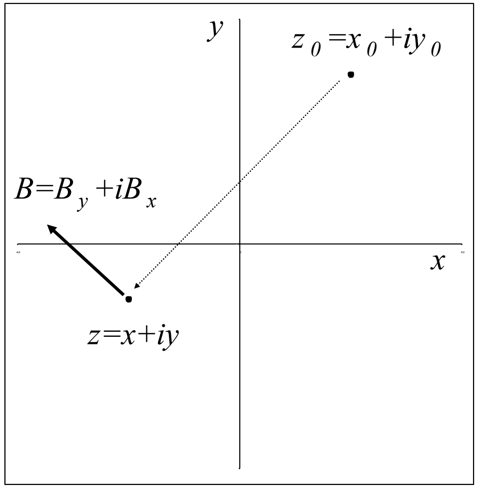
FIELD OF A CURRENT LINE: COMPLEX NOTATION

- Field given by a current line (Biot-Savart law)
  - Infinite current line

$$\vec{B}(\vec{r}) = \frac{\mu_0}{2\pi}\frac{\vec{I}\times\vec{r}}{|r|^2}$$

$$B_x(x,y) = -\frac{\mu_0 I}{2\pi}\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2},$$

$$B_y(x,y) = \frac{\mu_0 I}{2\pi}\frac{x-x_0}{(x-x_0)^2+(y-y_0)^2},$$



  - Complex notation

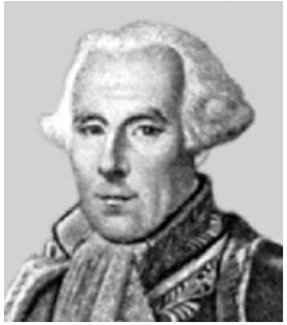
$$B_y(x,y) + \mathrm{i}B_x(x,y) = \frac{\mu_0 I}{2\pi}\frac{(x-x_0)-\mathrm{i}(y-y_0)}{(x-x_0)^2+(y-y_0)^2}$$

  - Using the relation

$$\frac{a-\mathrm{i}b}{a^2+b^2} = \frac{a-\mathrm{i}b}{(a+\mathrm{i}b)(a-\mathrm{i}b)} = \frac{1}{(a+\mathrm{i}b)}$$

  - We obtain the compact very useful notation

$$B(z) = \frac{\mu_0 I}{2\pi(z-z_0)}$$



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ANOTHER DIGRESSION ON TAYLOR SERIES

- In Unit 1 we saw that, for ε<1 one can write

$$(1+\varepsilon)^{\alpha} = 1+\alpha\varepsilon+O(\varepsilon^2)$$

- This is an extremely useful equation
  - Example: 1/(1+0.1) =1/1.1=0.90909 but using the approximation$(1+0.1)^{-1}$=1-0.1+O(0.01) =0.9 and I neglect something that is order of 0.01
- Now for *a*=-1 the whole series can be written
- Example

$$\frac{1}{1-t} = 1+t+t^2+t^3+\ldots = \sum_{n=1}^{\infty} t^{n-1}$$

$$\frac{1}{1-1/2} = \frac{1}{1/2} = 2$$

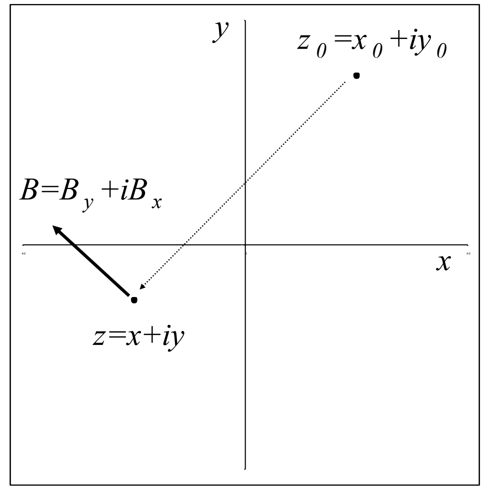
$$\frac{1}{1-1/2} = 1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\ldots = 1+0.5+0.25+0.125+\ldots = 1.875+O(0.06)$$

FIELD HARMONICS OF A CURRENT LINE

- Field given by a current line (Biot-Savart law)

$$B(z) = B_y(z) + iB_x(z)$$

$$B(z) = \frac{I\mu_0}{2\pi(z - z_0)} = -\frac{I\mu_0}{2\pi z_0}\frac{1}{1 - \frac{z}{z_0}}$$



using

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \ldots = \sum_{n=1}^{\infty} t^{n-1} \qquad |t| < 1 \text{ !!!}$$

we get

$$B(z) = -\frac{I\mu_0}{2\pi z_0}\sum_{n=1}^{\infty}\left(\frac{z}{z_0}\right)^{n-1} = -\frac{I\mu_0}{2\pi z_0}\sum_{n=1}^{\infty}\left(\frac{R_{ref}}{z_0}\right)^{n-1}\left(\frac{x+iy}{R_{ref}}\right)^{n-1}$$

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FIELD HARMONICS OF A CURRENT LINE

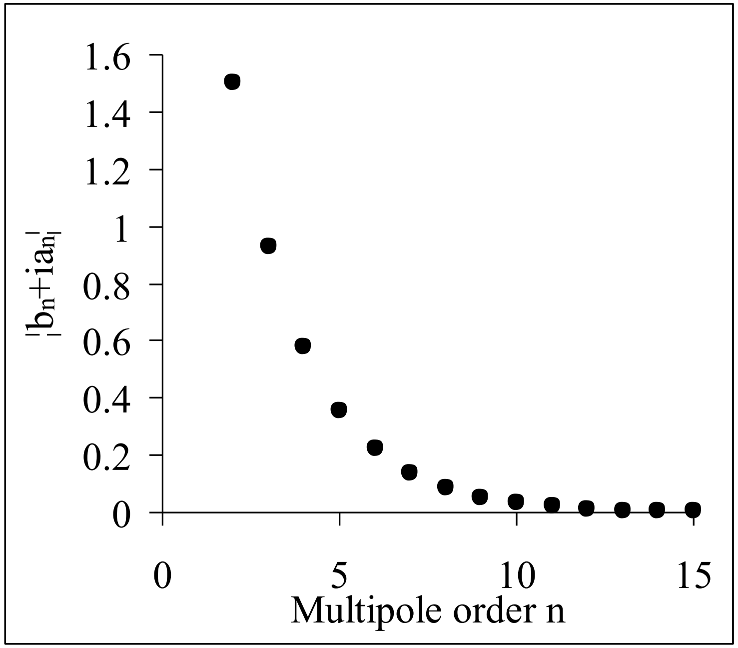
- Now we can compute the multipoles of a current line at z_0

$$B(z) = -\frac{I\mu_0}{2\pi z_0}\sum_{n=1}^{\infty}\left(\frac{z}{z_0}\right)^{n-1} = -\frac{I\mu_0}{2\pi z_0}\sum_{n=1}^{\infty}\left(\frac{R_{ref}}{z_0}\right)^{n-1}\left(\frac{x+iy}{R_{ref}}\right)^{n-1} \qquad |x+iy| < |z_0|$$

$$B_y + iB_x = 10^{-4} B_1 \sum_{n=1}^{\infty}(b_n + ia_n)\left(\frac{x+iy}{R_{ref}}\right)^{n-1}$$

$$B_1 = -\frac{I\mu_0}{2\pi}\mathrm{Re}\left(\frac{1}{z_0}\right)$$

$$b_n + ia_n = -\frac{I\mu_0 10^4}{2\pi z_0 B_1}\left(\frac{R_{ref}}{z_0}\right)^{n-1}$$

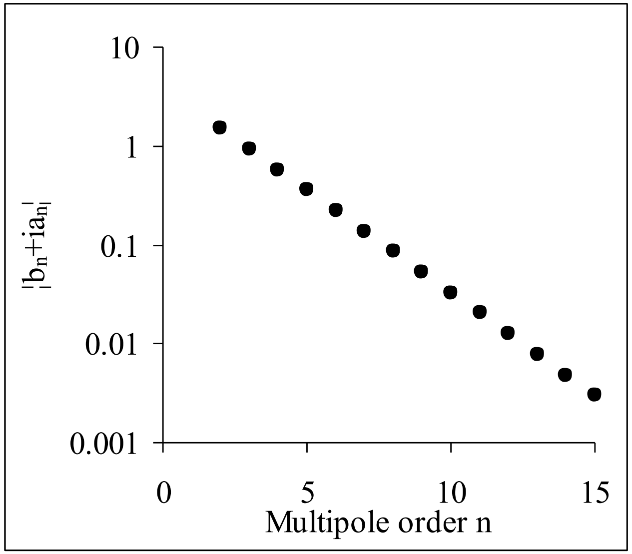


FIELD HARMONICS DECAY OF A CURRENT LINE

- Multipoles given by a current line decay with the order

$$b_n + ia_n = -\frac{I\mu_0 10^4}{2\pi z_0 B_1}\left(\frac{R_{ref}}{z_0}\right)^{n-1}$$

$$\ln\left(\left|b_n + ia_n\right|\right) = \ln\left(\frac{|I|\mu_0 10^4}{2\pi R_{ref} B_1}\right) + n\ln\left(\frac{R_{ref}}{|z_0|}\right)$$



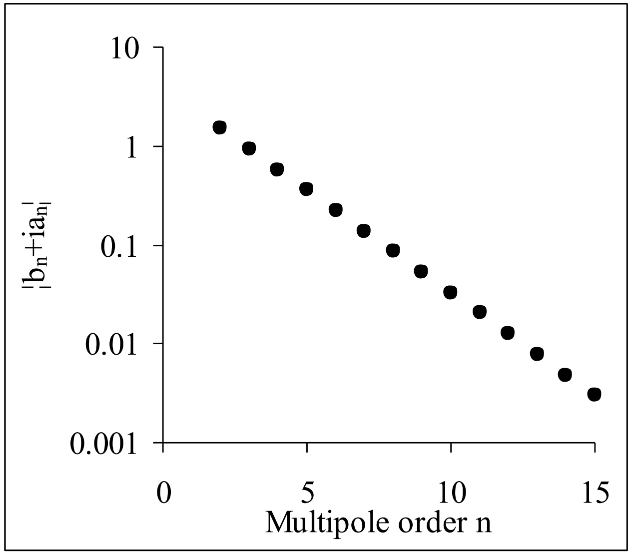
- The slope of the decay is the logarithm of $(R_{ref}/|z_0|)$
  - At each order, the multipole decreases by a factor $R_{ref}/|z_0|$
  - The decay of the multipoles tells you the ratio $R_{ref}/|z_0|$, i.e. where is the coil w.r.t. the reference radius –
  - like a radar … one can detect assembly errors in the magnet through slope of the decay of the anomalies of the magnetic field shape (we will come back to this)

FIELD HARMONICS DECAY OF A CURRENT LINE

- Multipoles given by a current line decay with the order

$$b_n + ia_n = -\frac{I\mu_0 10^4}{2\pi z_0 B_1}\left(\frac{R_{ref}}{z_0}\right)^{n-1}$$

$$\ln\left(\left|b_n + ia_n\right|\right) = \ln\left(\frac{\left|I\right|\mu_0 10^4}{2\pi R_{ref} B_1}\right) + n\ln\left(\frac{R_{ref}}{\left|z_0\right|}\right)$$



- The semilog scale is the natural way to plot multipoles
  - This is the point of view of Biot-Savart
- But usually specifications are on a linear scale
  - In general, multipoles must stay below one or a fraction of units – see later
  - This explains why only low order multipoles, in general, are relevant

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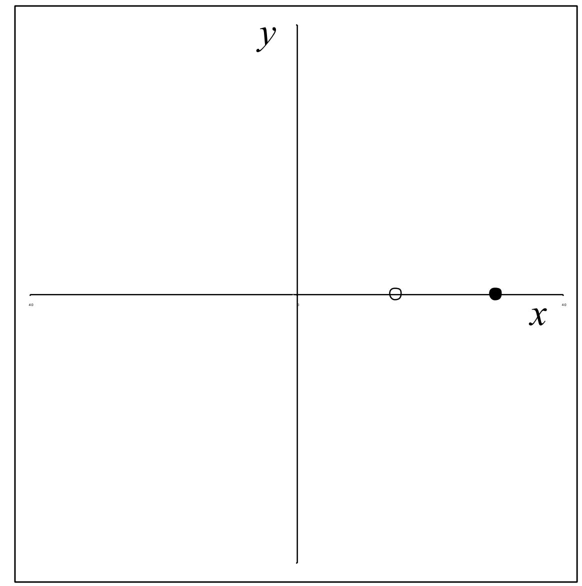
VALIDITY LIMITS OF FIELD HARMONICS

- When we expand a function in a power series we lose something

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \ldots = \sum_{n=1}^{\infty} t^{n-1} \qquad |t| < 1$$



  - Example 1. In t=1/2, the function is

$$\frac{1}{1-1/2} = \frac{1}{1/2} = 2$$

and using the series one has

$$1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots = 1 + 0.5 + 0.25 + 0.125 = 1.875 + \ldots$$

not bad … with 4 terms we compute the function within 7%

- When we expand a function in a power series we lose something

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \ldots = \sum_{n=1}^{\infty} t^{n-1} \qquad |t| < 1$$

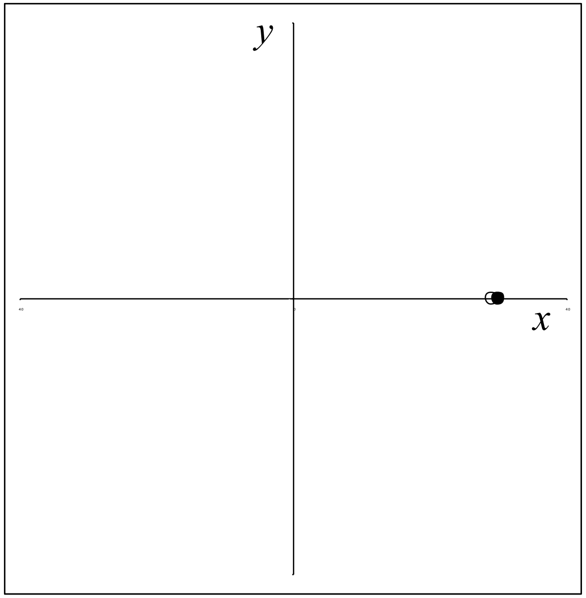
  - Ex. 2. In t=1, the function is infinite

$$\frac{1}{1-1} = \frac{1}{0} = \infty$$

and using the series one has

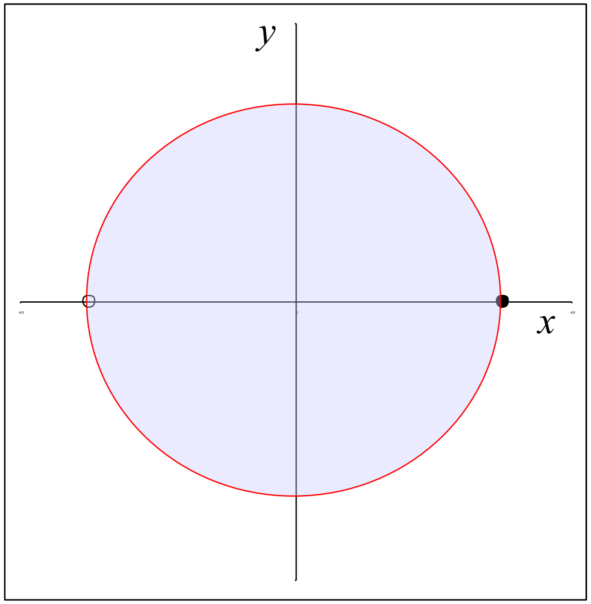
$$1 + 1 + (1)^2 + (1)^3 + \ldots = 1 + 1 + 1 + 1 + \ldots$$

which diverges … this makes sense



- When we expand a function in a power series we lose something

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \ldots = \sum_{n=1}^{\infty} t^{n-1} \qquad |t| < 1$$



- Ex. 3. In t=-1, the function is well defined

$$\frac{1}{1+1} = \frac{1}{2}$$

BUT using the series one has

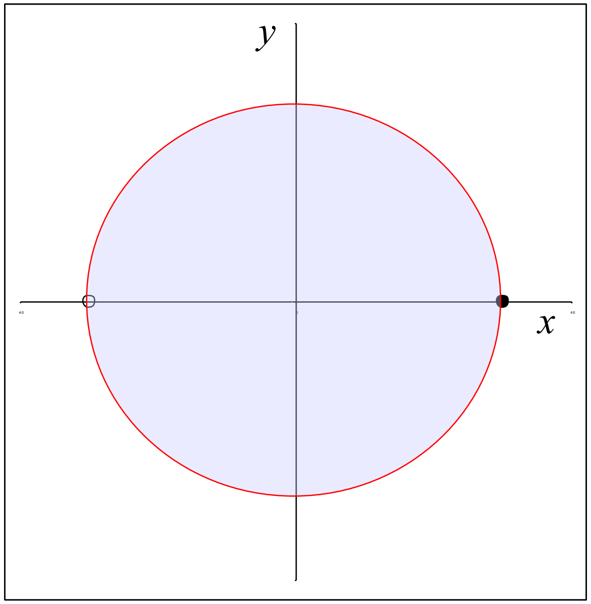
$$1 + (-1) + (-1)^2 + (-1)^3 + \ldots = 1 - 1 + 1 - 1 + \ldots$$

even if the function is well defined, the series does not work: we are outside the convergence radius

A GLIMPSE ON DIVERGENT SERIES

- If we are very clever, we can resum a divergent series

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \ldots = \sum_{n=1}^{\infty} t^{n-1} \qquad |t| < 1$$



- Ex. 4. In t=-2, the function is well defined

$$\frac{1}{1+2} = \frac{1}{3}$$

BUT using the series one has

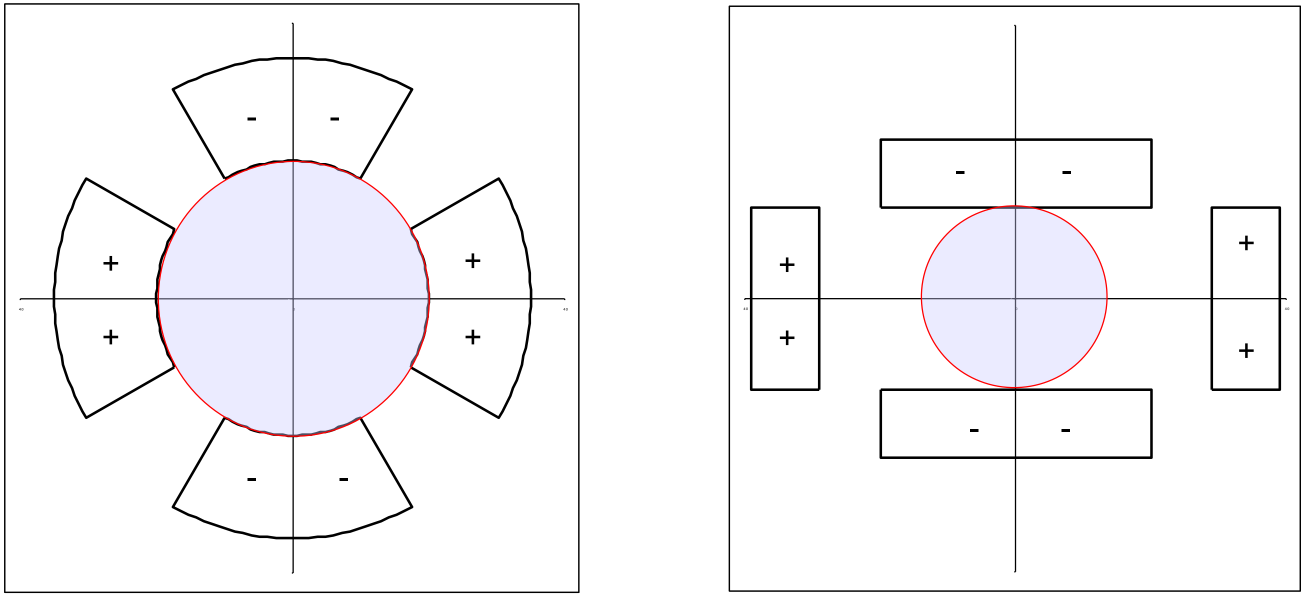
$$1 + (-2) + (-2)^2 + (-2)^3 + \ldots = 1 - 2 + 4 - 8 + \ldots$$

If I am able to recognize this, I can resum 1-2+4-8 + … = 1/3

This happens if you made an expansion to solve a problem and you are using it outside the series validity limits – there are several type of divergent series and ways to renormalize, remove singularities, etc - this is just one type

VALIDITY LIMITS OF FIELD HARMONICS

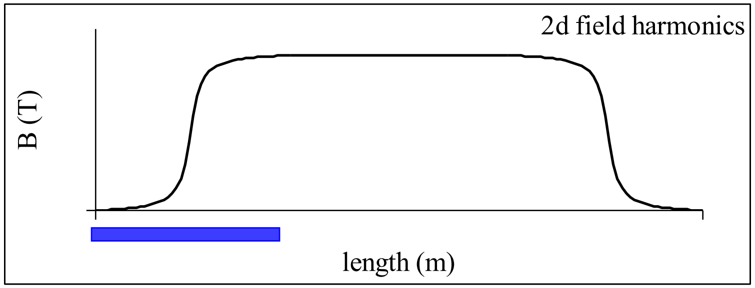
- If we have a circular aperture, the field harmonics expansion relative to the center is valid within the aperture

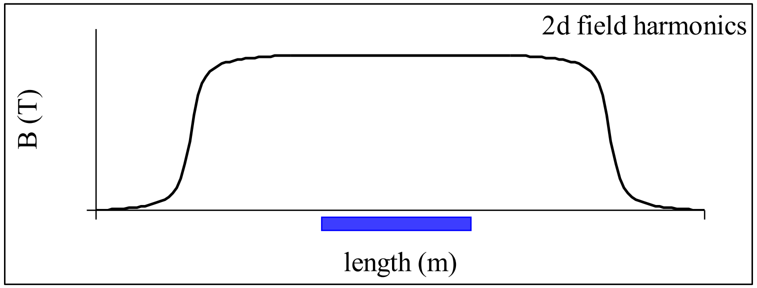


- For other shapes, the expansion is valid over a circle that touches the closest current line
- Don't use multipoles to compute the field in the coil !!

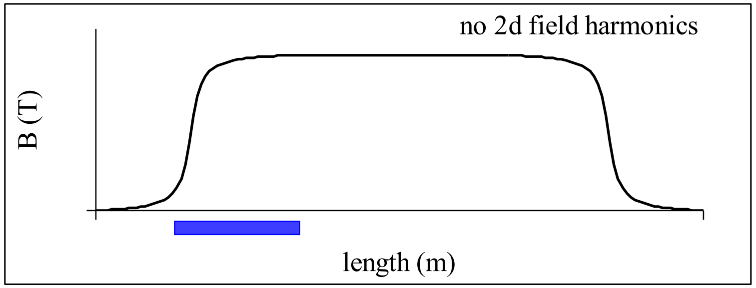
VALIDITY LIMITS OF FIELD HARMONICS

- Field harmonics in the heads
  - Harmonic measurements are done with rotating coils of a given length (see unit 21) – they give integral values over that length
    - If the rotating coil extremes are in a region where the field does not vary with z, one can use the 2d harmonic expansion for the integral





    - If the rotating coil extremes are in a region where the field vary with z, one cannot use the 2d harmonic expansion for the integral
    - One has to use a more complicated expansion



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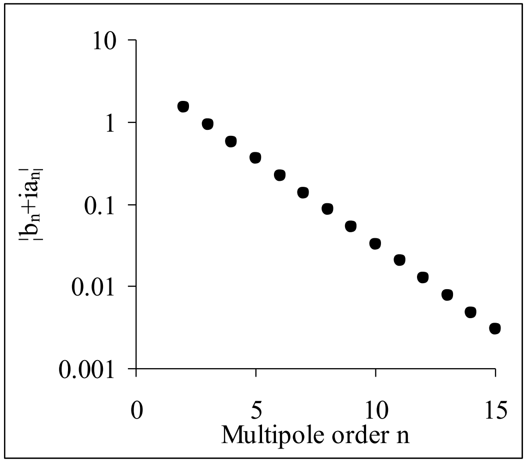
- Definition of field harmonics
- Field harmonics of a current line
- Validity limits of field harmonics
- Some applications of tough mathematics to magnetic field measurements
- Hints on beam dynamics requirements on field harmonics

FIRST APPLICATION: VERIFICATION OF A MEASUREMENT SYSTEM

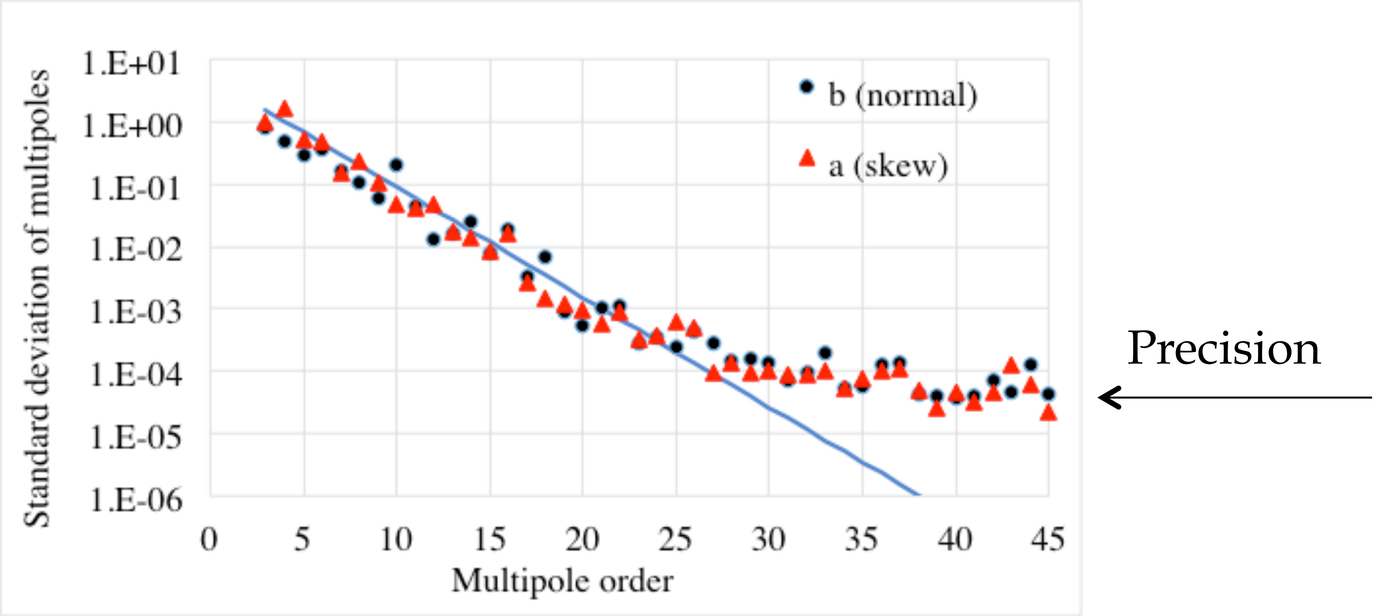
- The decay of multipoles is a powerful tool to verify the consistency of a magnetic measurement
  - Let us take a measure of a magnetic field of a magnet via rotating coils
  - Let us assume we have N consecutive measurements along the magnet axis
    - $b_n(k)$, $a_n(k)$ k=1, 2, N
  - We compute the standard deviation of each multipole and we plot in a semilog scale
    - If the measurement is well done and reference radius is 2/3 of the aperture the slope is 2/3
    - This means that the stdev of every successive multipole is 2/3 the previous one
    - Every two orders you reduce by a factor two



What is expected as multipole decay

SECOND APPLICATION: VERIFICATION OF A MEASUREMENT SYSTEM

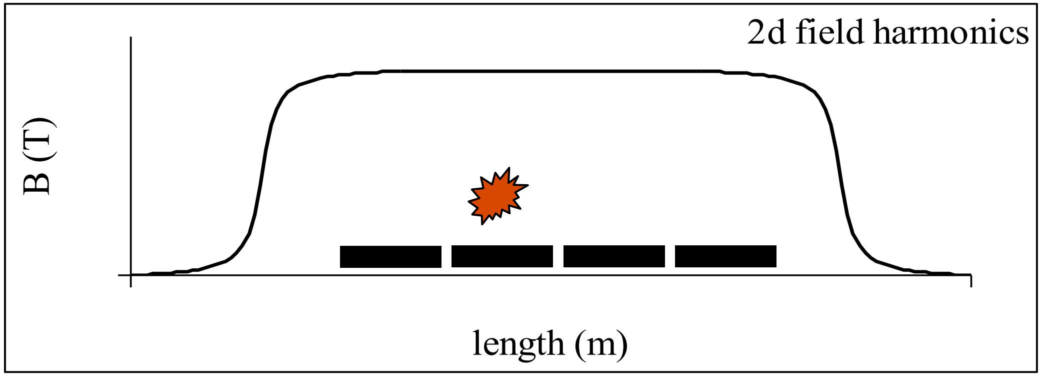
- … and this is a real example from the HL_LHC triplet
  - Not only we check that the measurement is ok …
  - … but the place where the line starts getting horizontal is the precision of the measurement system – here we have a precision of 10^{-4} units, that means 10^{-8} of the main field (0.01 ppm)

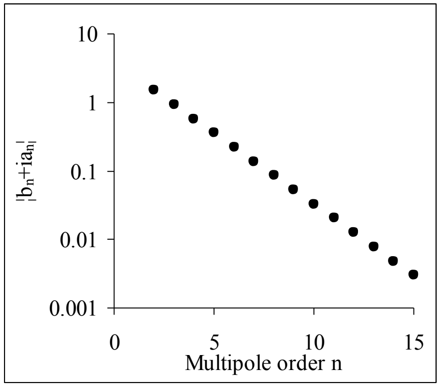


Decay of standard deviation of multipoles measured in different sections of MQXFB quadrupole
(L. Fiscarelli, P. Rogacki)

THIRD APPLICATION: FINDING ASSEMBLY ERRORS

- Another example: we have a localized assembly error
  - We have four measurements, one of them is affected by the error
    - We compute the difference between the anomaly (measurements 2) and the average of 1, 3, and 4
  - We put the result in semilog scale
    - The slope multiplied by the reference radius will give the distance of the assembly error
    - If error is far from the aperture, slope is larger (decay is more rapid, that means it will be visible only on low order multipoles





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BEAM DYNAMICS REQUIREMENTS

- Main component of the dipoles (field)
  - This is ensuring the orbit along the ring
    - Typically a absolute knowledge of the magnetic field within 0.1%
    - A spread between magnets of the order of 0.1%
    - A reproducibility of the order of 0.01%
  - We will see that the tolerances needed for building superconducting magnets naturally guarantee these levels
    - Cables are positioned within 0.05 mm, apertures of the order of 25 mm, this gives 0.2% error in field, and similar values for spread
    - For reproducibility it is critical to cycle the magnets

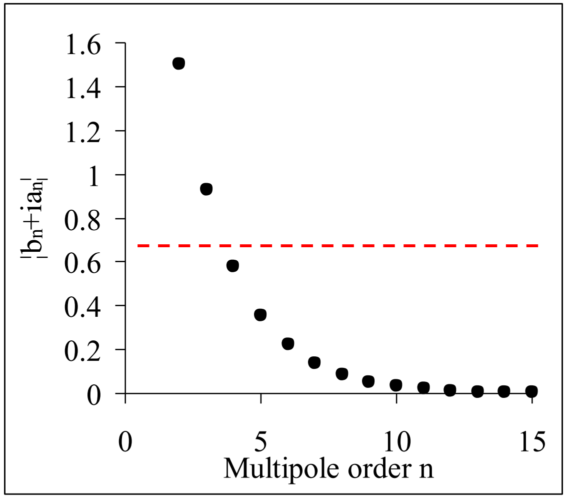
BEAM DYNAMICS REQUIREMENTS

- Main component of the quadrupoles (gradient)
  - The tune has to be controlled within 0.001
  - Tune is proportional to quadrupole main component
  - If the total tune is large (for instance 60 in the LHC) even 0.01% variation of quadrupole force is visible
    - Corrector elements and feedback solve the problem – magnets alone cannot reach this level
  - In general a absolute knowledge of quadrupole main component within 0.2% is achievable, a spread of 0.1%, and a reproducibility of 0.01%
    - Note that accelerators with larger number of cells (larger tune) become more difficult

BEAM DYNAMICS REQUIREMENTS

- Sextupolar components
  - Sextupole gives chromaticity – to be controlled on a edge of a cliff (positive but smaller than 10, negative values make the beam unstable)
  - Dipoles have sextupolar components that needs to be controlled within 0.1 units (normalized multipoles)
    - Example: in the LHC 1 units of b_3 in the dipoles give 40 chromaticity units – so one need to know and control b_3 within 0.05 units,
  - This is within reach of measurement systems, and with proper precycling reproducibility can be guaranteed

- High order mutipoles
  - Rule of thumb (just to give a zero order idea): field harmonics have to be of the order of 0.1 to 1 unit
    - Higher order are ignored in beam dynamics codes (in LHC up to order 11 only)
    - Note that spec is rather flat, but multipoles are decaying !! – Therefore in principle higher orders cannot be a problem



SUMMARY

- We outlined the Maxwell equations for the magnetic field
  - They give a large constraint on the shape of the magnetic field

$$\nabla \cdot B = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \qquad \nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$$

- We have seen that for a long magnet we can express the transverse field inside the aperture with a series of multipoles

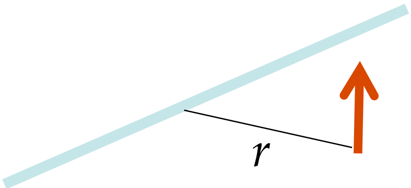
$$B_y + iB_x = 10^{-4} B_1 \sum_{n=1}^{\infty} (b_n + ia_n) \left(\frac{x+iy}{R_{ref}} \right)^{n-1}$$

  - Compact way of representing the field
  - Biot-Savart: multipoles decay with multipole order as a power law
  - Attention !! Validity limits and convergence domains

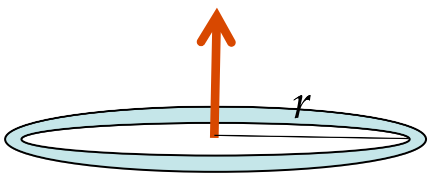
SUMMARY

- We have seen how magnetic field is generated by a current line

$$\vec{B}(\vec{r}) = \frac{\mu_0}{2\pi} \frac{\vec{I} \times \vec{r}}{|r|^2}$$



$$B = \frac{\mu_0 I}{2r}$$

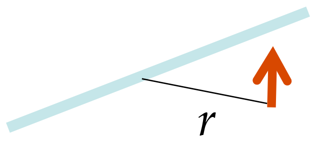


  - We have seen that μ is terribly small, making our work very challenging

SUMMARY

- We have computed the multipoles of a current line

$$\vec{B}(\vec{r}) = \frac{\mu_0}{2\pi} \frac{\vec{I} \times \vec{r}}{|r|^2}$$



  - Using complex numbers it is quite fast …

$$\boxed{B(z) = \frac{\mu_0 I}{2\pi(z - z_0)}}$$

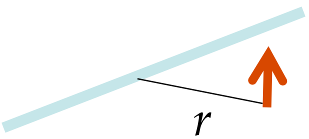
  - …as long as you know that in a certain region (inside the magnet aperture, far from current lines)

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \ldots = \sum_{n=1}^{\infty} t^{n-1}$$

SUMMARY

- We have computed the multipoles of a current line

$$\vec{B}(\vec{r}) = \frac{\mu_0}{2\pi}\frac{\vec{I}\times\vec{r}}{|r|^2}$$



  - Multipoles of a current line decay as a power law

$$b_n + i a_n = -\frac{I\mu_0 10^4}{2\pi z_0 B_1}\left(\frac{R_{ref}}{z_0}\right)^{n-1}$$

  - Therefore:
    - Increasing order they become so small that you can neglect them and stop the computation (order 10 to 15)
    - Magnet specification are until order 10-15
    - Magnet optimization can stop at order 10-15
    - The measurement of field anomalies tell you where is the problem (the distance, the angle is more complicated)

COMING SOON

- Coming soon …
  - It is useful to have magnets that provide pure field harmonics
  - How to build a pure field harmonic (dipole, quadrupole …) [pure enough for the beam …] with a cable ? Which field/gradient can be obtained ?

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  - Hardy, "Divergent series", first chapter (don't go further)

- On the field model
  - A.A.V.V. "LHC Design Report", CERN 2004-003 (2004) pp. 164-168.
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- A. Jain for discussions about reference radius, multipoles in heads, vector potential
- S. Russenschuck for discussions about vector potential
- G. Turchetti for teaching me analytic functions and divergent series, and other complicated subjects in a simple way
- www.wikipedia.org for most of the pictures

DEFINITION OF FIELD HARMONICS

- Vector potential
  - Since $\nabla \cdot B = 0$ one can always define a vector potential $\boldsymbol{A}$ such that

$$\nabla \times A = B$$

  - The vector potential is not unique (gauge invariance): if we add the gradient of any scalar function, $A' = A + \nabla f$ it still satisfies

$$\nabla \times A' = \nabla \times A + \nabla \times \nabla f = \nabla \times A = B$$

- Scalar potential
  - In the regions free of charge and magnetic material $\nabla \times B = 0$

    Therefore in this case one can also define a scalar potential (such as for gravity) $-\nabla g = B$

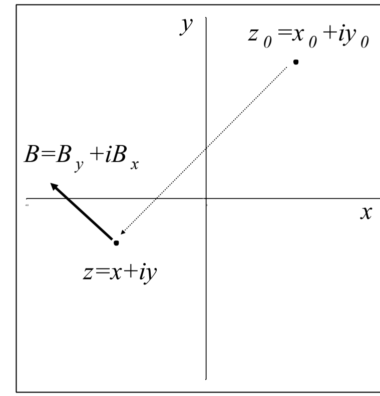
- One can prove that $A + ig$ is an analytic function in a region free of charge and magnetic material

FIELD HARMONICS OF A CURRENT LINE

- Field given by a current line (Biot-Savart law) – vector potential formalism …

$$B_\theta = -\frac{\partial A_z}{\partial r} \qquad B_r = \frac{1}{r}\frac{\partial A_z}{\partial \theta}$$

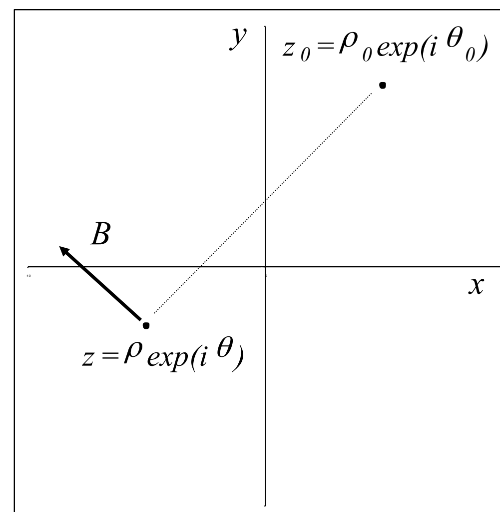
$$A_z(\rho,\theta) = -\frac{\mu_0 I}{2\pi}\ln\left(\frac{R}{\rho_0}\right) = -\frac{\mu_0 I}{2\pi}\ln\left(\frac{\sqrt{\rho_0^2+\rho^2-2\rho_0\rho\cos(\theta-\theta_0)}}{\rho_0}\right)$$



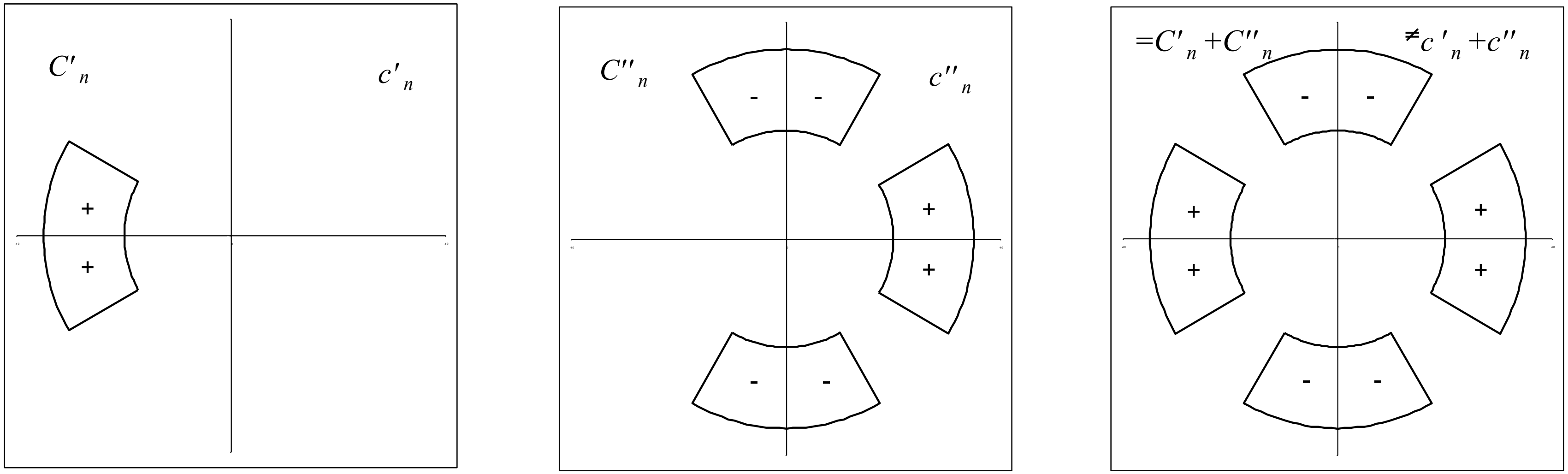
… and polar coordinates formalism

$$B_r = -\frac{\mu_0 I}{2\pi\rho_0}\sum_{n=1}^{\infty}\left(\frac{\rho}{\rho_0}\right)^{n-1}\sin[n(\theta-\theta_0)]$$

$$B_\theta = -\frac{\mu_0 I}{2\pi\rho_0}\sum_{n=1}^{\infty}\left(\frac{\rho}{\rho_0}\right)^{n-1}\cos[n(\theta-\theta_0)]$$



FIELD HARMONICS: LINEARITY



- Linearity of coefficients (very important)
  - Non-normalized coefficients are additive
  - Normalized coefficients are not additive

$$c_n = \frac{C_n}{B_1} = \frac{C_n^{'} + C_n^{''}}{B_1^{'} + B_1^{''}} \neq \frac{C_n^{'}}{B_1^{'}} + \frac{C_n^{''}}{B_1^{''}} = c_n^{'} + c_n^{''}$$

  - Normalization gives handy (and physical) quantities, but some drawbacks – pay attention !!